LAST TIME :

$$\int_C \underline{F} \cdot d\underline{r}$$

VECTOR LINE INTEGRAL ALONG AN ORIENTED CURVE C.

HOW TO CALCULATE :

1) PARAMETRIZE THE CURVE

$$\underline{r}(t), \quad a \le t \le b$$

SUCH THAT $\underline{r}'(t)$ IS NON-ZERO

AND POINTS IN +VE DIRECTION.

2) $$\int_C \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt .$$

MEANING : $$\int_C \underline{F} \cdot d\underline{r} = \int_C \underline{F} \cdot \underline{T}\, ds$$

WHERE $\underline{T}$ IS A VECTOR FIELD OF UNIT LENGTH POINTING IN THE +VE DIRECTION ALONG C.

= WORK DONE BY A FORCE FIELD $\underline{F}$ ON A PARTICLE MOVING ALONG THE CURVE C.

EVENTUALLY : GIVEN A VECTOR FIELD $\underline{F}$ AND A POINT P.

LET $D(P)$ BE A DISK AROUND P,
$\partial D(P)$ ITS BOUNDARY,
$\underline{n}$ THE UNIT NORMAL TO $D(P)$ AT P.

THEN

$$\lim_{\substack{D(P) \text{ SHRINKS} \\ \text{TO } P}} \frac{\displaystyle\int_{\partial D(P)} \underline{F} \cdot d\underline{r}}{\text{AREA}\big(D(P)\big)} = \left(\nabla \times \underline{F}\right) \cdot \underline{n} .$$

TODAY : THE FUNDAMENTAL THM OF CALCULUS AND PATH-INDEPENDENT VECTOR FIELDS.

NOTATION : WHEN C IS CLOSED, THE LINE INTEGRAL IS OFTEN CALLED THE <u>CIRCULATION OF $\underline{F}$ AROUND C.</u> USE

$$\oint_C \underline{F} \cdot d\underline{r} .$$

DEFN : SUPPOSE $\underline{F}$ IS A VECTOR FIELD DEFINED ON SOME DOMAIN D.

WE SAY $\underline{F}$ IS <u>PATH-INDEPENDENT ON D</u> IF

$$\int_C \underline{F} \cdot d\underline{r}$$

DEPENDS ONLY ON THE START AND ENDPOINT OF C.

NON-EXAMPLE : LET $\underline{F}(x, y, z) = \dfrac{(-y, x, 0)}{x^2+y^2}$

BE THE <u>VORTEX FIELD</u>.

ITS DOMAIN IS $\mathbb{R}^3 - (z\text{-AXIS})$.

LET $\underline{r}(t) = (\cos t, \sin t, 0)$.

LETTING t RANGE FROM 0 TO t_0 GIVES A CURVE C_{t_0}.

$$\int_{C_{t_0}} \underline{F} \cdot d\underline{r} = t_0$$

(WE BASICALLY DID THIS CALCULATION LAST TIME)

FOR ANY t_0, C_{t_0} AND $C_{t_0 + 2\pi}$

HAVE THE SAME START AND ENDPOINT BUT INTEGRATE TO GIVE DIFFERENT VALUES.

THE VORTEX FIELD IS <u>NOT</u> PATH-INDEPENDENT ON $\mathbb{R}^3 - (z\text{-AXIS})$.

REMARK: PATH-INDEPENDENCE OF A VECTOR FIELD $\underline{F}$ IMPLIES THAT THE CIRCULATION OF $\underline{F}$ IS ALWAYS 0.

THUS: WE CAN SEE THE VORTEX FIELD IS NOT PATH-INDEPENDENT ON $\mathbb{R}^3 - (z\text{-AXIS})$ SINCE

$$\oint_{C_{2\pi}} \underline{F} \bullet d\underline{r} = 2\pi \neq 0.$$

RECALL : FTC I

SUPPOSE f IS CONTINUOUSLY DIFFERENTIABLE ON $[a,b]$,

THEN

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

THM : IF C IS AN ORIENTED CURVE FROM P TO Q, & f IS CONT. DIFF. ON A DOMAIN D CONTAINING C, THEN

$$\int_C \nabla f \cdot d\underline{r} = f(Q) - f(P).$$

COR : IF f IS CONT. DIFF. ON A DOMAIN D, THEN ∇f IS PATH-INDEPENDENT ON D.

RECALL : FTC II

SUPPOSE F IS CONTINUOUS ON $[a,b]$.

DEFINE $f(x) := \int_a^x F(t)\, dt$.

THEN $f'(x) = F(x)$.

THM : SUPPOSE D IS A PATH-CONNECTED DOMAIN, $\underline{F}$ IS PATH-INDEPENDENT ON D.

FIX $P_0 \in D$, DEFINE $f(P) := \int_C \underline{F} \cdot d\underline{r}$

WHERE C IS AN ORIENTED CURVE FROM P_0 TO P.

THEN $\nabla f = \underline{F}$.

COR : $\underline{F}$ IS PATH-INDEPENDENT ON A DOMAIN D

IF AND ONLY IF

THERE IS A FUNCTION f WITH DOMAIN D

SUCH THAT $\nabla f = \underline{F}$.

IN THIS CASE, f IS CALLED A POTENTIAL FOR $\underline{F}$, AND $\underline{F}$ IS ALSO SAID TO BE <u>CONSERVATIVE ON D</u>.

NOTE: SINCE CURL OF A GRADIENT IS 0,

CONSERVATIVE VECTOR FIELDS MUST HAVE CURL = 0.

EXAMPLES:

1) $\underline{F}(x,y,z) = (0,x,0)$.

$\nabla \times \underline{F} = (0,0,1) \neq 0$

SO $\underline{F}$ IS NOT CONSERVATIVE (ON $\mathbb{R}^3$).

2) $\underline{F}(x,y,z) = \dfrac{(y,-x,0)}{x^2+y^2}$.

$\nabla \times \underline{F} = 0$.

HOWEVER, $\displaystyle\oint_{\substack{x^2+y^2=1,\\ z=0}} \underline{F} \cdot d\underline{r} = 2\pi \neq 0.$

a) $\underline{F}$ IS NOT CONSERVATIVE ON $\mathbb{R}^3 - (z\text{-AXIS})$.

b) $\underline{F}$ IS CONSERVATIVE ON $\mathbb{R}^3 - (yz\text{-PLANE})$.

$f(x,y,z) = \arctan\left(\frac{y}{x}\right)$ IS A POTENTIAL FOR $\underline{F}$.

3) $\underline{F}(x,y,z) = \dfrac{(x,y)}{\sqrt{x^2+y^2}}$ IS CONSERVATIVE ON $\mathbb{R}^3 - (z\text{-AXIS})$.

TEST FOR CONSERVATIVE OR NOT

1) CURL $\neq 0$ $\Rightarrow$ NOT CONSERVATIVE

2) CAN FIND A NON-ZERO CIRCULATION

$\Rightarrow$ NOT CONSERVATIVE

N.B. WHEN CURL = 0
CAN ONLY HOPE TO ACHIEVE
THIS IF DOMAIN HAS HOLES
TO WRAP AROUND

3) SEEK OUT A POTENTIAL

$\Rightarrow$ CONSERVATIVE.