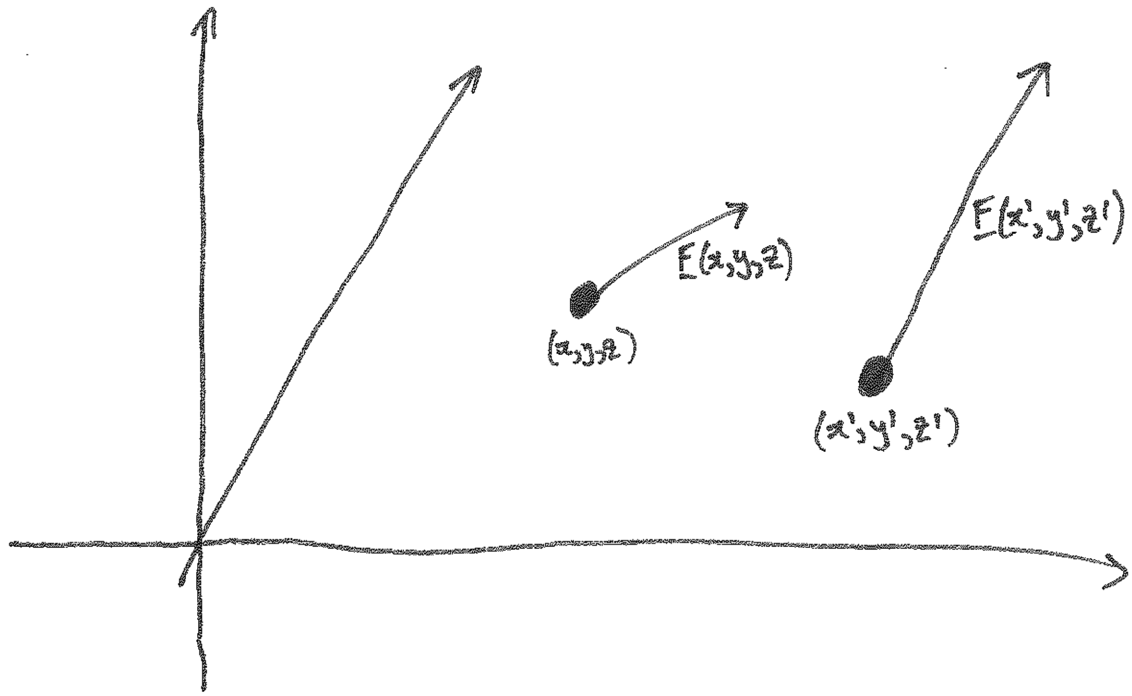
# DIV, GRAD, CURL, AND ALL THAT ...

A VECTOR FIELD $\underline{F}$ ASSIGNS TO EACH POINT $(x, y, z)$
A VECTOR $\underline{F}(x, y, z)$




YOU CAN THINK OF THIS AS A COLLECTION OF ARROWS WHICH VARY FROM POINT TO POINT.

THEY DESCRIBE THINGS LIKE WIND VELOCITY. OR HOW A FLUID MOVES.

RECALL $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$.

ON THE ONE HAND, IT IS WRONG TO THINK OF THIS AS A VECTOR. ON THE OTHER, IT IS USEFUL FOR REMEMBERING FORMULAE.

GIVEN A FUNCTION $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z)$

$$\nabla f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix}$$

IS A VECTOR FIELD, THE <u>GRADIENT</u> OF $f$.

THIS VECTOR FIELD IS ORTHOGONAL TO THE LEVEL CURVES OF $f$, POINTS IN THE DIRECTION OF MAXIMUM RATE OF INCREASE, AND THE MAX RATE OF INCREASE IS GIVEN BY $\|\nabla f\|$.

THE CURL OF A VECTOR FIELD

$$\underline{F}(x,y,z) = \left(F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z)\right)$$

IS GIVEN BY

$$\nabla \times \underline{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

A NEW VECTOR FIELD.

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THE DIVERGENCE OF A VECTOR FIELD

$$\underline{F}(x,y,z) = \left(F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z)\right)$$

IS GIVEN BY

$$\nabla \cdot \underline{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

A FUNCTION $\mathbb{R}^3 \to \mathbb{R}$.

$$\text{FUNCTIONS} \xrightarrow{\text{GRAD}} \text{VECTOR FIELDS} \xrightarrow{\text{CURL}} \text{VECTOR FIELDS} \xrightarrow{\text{DIV}} \text{FUNCTIONS}$$

SOME IDENTITIES:

"CURL OF A GRADIENT IS 0":

$$\nabla \times \nabla f = \nabla \times \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \\ \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \end{pmatrix} = 0.$$

"DIVERGENCE OF A CURL IS 0":

$$\nabla \cdot \left( \nabla \times \underline{F} \right) = 0$$ (HOMEWORK).

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"DIVERGENCE OF THE GRADIENT IS THE LAPLACIAN"

$$\nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$$
$$\| \quad \nabla^2 f$$

(THIS SHOWS UP IN HEAT AND WAVE EQUATION. ITS EIGENVALUES RELATE TO THE SOUND OF AN INSTRUMENT)

"OTHER"

$$\nabla \left( \nabla \cdot \underline{F} \right) - \nabla \times \left( \nabla \times \underline{F} \right) = \begin{pmatrix} \nabla^2 F_1 \\ \nabla^2 F_2 \\ \nabla^2 F_3 \end{pmatrix}$$

SUPPOSE

$$\underline{F}(x, y, z) = \left(F_1(x, y),\ F_2(x, y),\ 0\right).$$

THEN

$$\nabla \times \underline{F} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \end{pmatrix},$$

$$\nabla \cdot \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

EXAMPLES : RADIAL 2D AND 3D

- WHEN $f(x,y,z) = r = \sqrt{x^2+y^2}$

$$\nabla f = \frac{(x,y,0)}{r}.$$

- WHEN $f(x,y,z) = \ln r$, $\nabla f = \dfrac{(x,y,0)}{r^2}$.
- WHEN $n \neq 2$, $f(x,y,z) = \dfrac{r^{2-n}}{2-n}$, $\nabla f = \dfrac{(x,y,0)}{r^n}$.

- IF $\underline{F}(x,y,z) = \dfrac{(x,y,0)}{r^n}$, $\nabla \times \underline{F} = \underline{0}$.
- IF $\underline{F}(x,y,z) = \dfrac{(x,y,0)}{r^2}$, $\nabla \cdot \underline{F} = 0$.

2 IS IMPORTANT!

SIMILARLY,

- WHEN $f(x,y,z) = \rho = \sqrt{x^2+y^2+z^2}$

$$\nabla f = \frac{(x,y,z)}{\rho}$$

- IF $\underline{F}(x,y,z) = \dfrac{(x,y,z)}{\rho^n}$, $\nabla \times \underline{F} = \underline{0}$
- IF $\underline{F}(x,y,z) = \dfrac{(x,y,z)}{\rho^3}$, $\nabla \cdot \underline{F} = 0$.

3 IS IMPORTANT!

EXAMPLES : 2D ROTATIONAL

LET $\underline{P}(x,y,z) = \left( \dfrac{xz}{r^n}, \dfrac{yz}{r^n}, 0 \right)$

$$\nabla \times \underline{P} = \begin{pmatrix} \dfrac{-y}{r^n} \\ \dfrac{x}{r^n} \\ \dfrac{\partial}{\partial x}\left(\dfrac{yz}{r^n}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{xz}{r^n}\right) \end{pmatrix}$$

$$= \frac{(-y, x, 0)}{r^n}.$$

IF $\underline{F}(x,y,z) = \dfrac{(-y,x,0)}{r^n}$, $\quad \nabla \cdot \underline{F} = 0$.

IF $\underline{F}(x,y,z) = \dfrac{(-y,x,0)}{r^2}$, $\quad \nabla \times \underline{F} = \underline{0}$.

2 IS IMPORTANT.