CHANGE OF COORDINATES

EXAMPLES

- POLAR COORDINATES

$$G(r,\theta) = (r\cos\theta, r\sin\theta)$$

- CYLINDRICAL

$$G(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$$

- SPHERICAL

$$G(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

QUESTION : HOW DO THESE COORDINATE CHANGES INTERACT WITH AREA AND VOLUME ?

ANSWER :

$$dA = r\,dr\,d\theta$$

$$dV = r\,dr\,d\theta\,dz$$

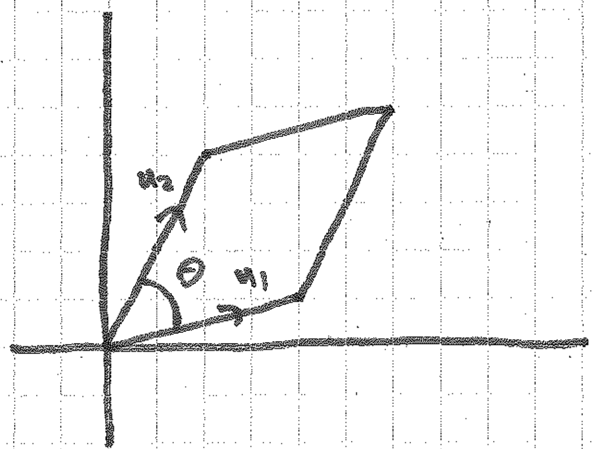
$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

MORE QUESTIONS

- WHY ???
- WHAT ABOUT $G(u,v) = (x(u,v), y(u,v))$, $G(u,v,w) = (x(u,v,w), \dots)$?

DETERMINANTS

1) AREA OF A PARALLELOGRAM



$$= |u_1||u_2||\sin\theta|$$

IF $u_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ $u_2 = \begin{pmatrix} b \\ d \end{pmatrix}$,

$$|u_1||u_2||\sin\theta|$$

$$= \left| \begin{pmatrix} a \\ c \\ 0 \end{pmatrix} \times \begin{pmatrix} b \\ d \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 \\ 0 \\ ad-bc \end{pmatrix} \right|$$

$$= |ad-bc|.$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad-bc.$$

2) VOLUME OF PARALLELOPIPED

$$= |(u_1 \times u_2) \cdot u_3|$$

$$\det \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} := (u_1 \wedge u_2) \cdot u_3.$$

(SIGN OF DETERMINANT RECORDS ORIENTATION).

LINEAR COORDINATE CHANGE

$$G(u,v) = (au+bv, cu+dv).$$

TAKES $(1,0) \rightsquigarrow (a,c)$

$(0,1) \rightsquigarrow (b,d)$.

AREA CHANGES BY $\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$.

ONLY CHECKED UNIT SQUARE.

BY CUTTING UP ARBITRARY DOMAINS INTO SQUARES
CAN CHECK IS TRUE FOR ALL DOMAINS.

SIMILARLY, $G(u,v,w) = (a_{11}u + a_{12}v + a_{13}w, a_{21}u + a_{22}v + a_{23}w, a_{31}u + a_{32}v + a_{33}w)$

CHANGES VOLUME BY

$$\left|\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right|.$$

JACOBIAN

NOTICE : IF $G(u,v) = (x(u,v), y(u,v))$
$$= (au + bv, cu + dv)$$

THEN $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ IS OBTAIN BY

DIFFERENTIATION $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$.

SIMILARLY, FOR 3D.

GIVEN AN ARBITRARY $G(u,v) = (x(u,v), y(u,v))$,

THE MATRIX $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ IS CALLED THE <u>JACOBIAN.</u>

$JAC(G)$.

WE HAVE HINTED THAT

$$\left| \det(\text{JACOBIAN}) \right|$$

RELATES AREA | VOLUME IN COORDINATE CHANGES.

$\boxed{\text{EXAMPLES}}$

1) WHEN $G(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$

$$J(G) = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det(J(G)) = r.$$

2) WHEN $G(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

$$J(G) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(J(G)) = r.$$

3) WHEN $G(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$$J(G) = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}$$

$$\det(J(G)) = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\rho\sin\phi\sin\theta \\ \rho\sin\phi\cos\theta \\ 0 \end{pmatrix} = \rho^2\sin\phi.$$

LAST TIME :

HAVE COORDINATE CHANGES

- $G(r,\theta) = (r\cos\theta,\ r\sin\theta)$ POLAR
- $G(r,\theta,z) = (r\cos\theta,\ r\sin\theta,\ z)$ CYLINDRICAL
- $G(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$ SPHERICAL

FOR A 2D COORDINATE CHANGE

$$G(u,v) = (x(u,v),\ y(u,v))$$

$$\text{JACOBIAN}(G) = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

FOR A 3D COORDINATE CHANGE

$$G(u,v,w) = (x(u,v,w),\ y(u,v,w),\ z(u,v,w))$$

$$\text{JACOBIAN}(G) = \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

STILL LAST TIME :

DETERMINANTS CALCULATE AREA / VOLUME OF PARALLELOGRAMS

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc .$$

$$\det \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} := (u_1 \times u_2) \cdot u_3 .$$

3D VECTORS

WE SAW FOR

POLAR $\det J(G) = r$

CYLINDRICAL $\det J(G) = r$

SPHERICAL $\det J(G) = \rho^2 \sin\phi$.

CHANGE OF VARIABLES FORMULA

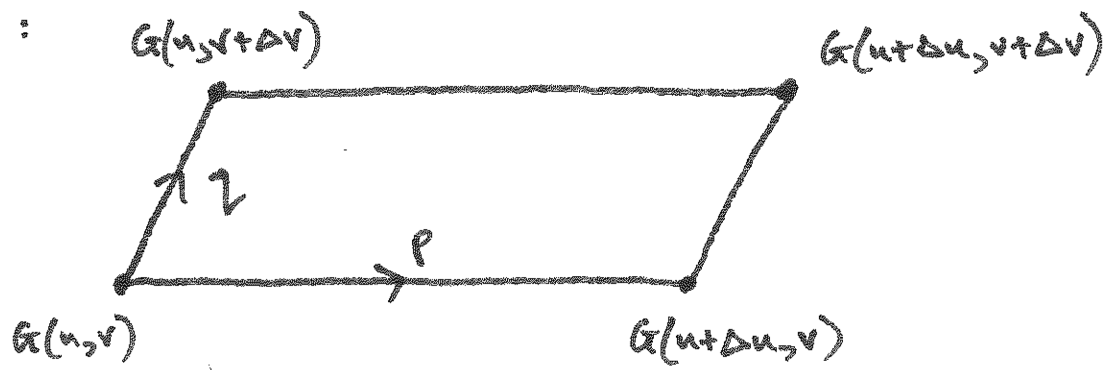
FROM THESE EXAMPLES GUESS

$$dA = \left| J(G) \right| \, du \, dv$$

OR

$$dV = \left| J(G) \right| \, du \, dv \, dw$$

"PROOF":



FOR SMALL $\Delta u, \Delta v$ THIS IS APPROXIMATELY A PARALLELOGRAM SO

AREA $\approx$ AREA OF PARALLELOGRAM W/ SIDES p, q.

$$p = G(u+\Delta u, v) - G(u,v) \approx \begin{pmatrix} x_u \\ y_u \end{pmatrix} \Delta u$$

$$q = G(u, v+\Delta v) - G(u,v) \approx \begin{pmatrix} x_v \\ y_v \end{pmatrix} \Delta v$$

SO

AREA $\approx$ AREA OF PARALLELOGRAM W/ SIDES

$$\begin{pmatrix} x_u \\ y_u \end{pmatrix} \Delta u, \quad \begin{pmatrix} x_v \\ y_v \end{pmatrix} \Delta v$$

$$= \left| \det \left(J(G) \right) \right| \Delta u \, \Delta v .$$

THM: SUPPOSE $G(u,v) = (x(u,v), y(u,v))$ IS A COORDINATE CHANGE (i.e. $\det J(G) \neq 0$ ALMOST EVERYWHERE)

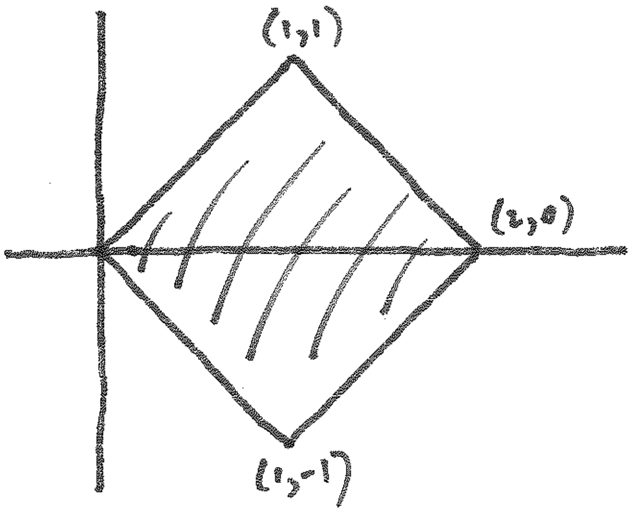
AND THAT G IS 1-1 ON (THE INTERIOR OF) A DOMAIN D.

THEN

$$\iint_{G(D)} f(x,y)\, dA = \iint_{D} f(x(u,v), y(u,v)) \left| \det J(G) \right| du\, dv.$$

SIMILARLY, IN 3D.

EXAMPLE



$$\iint_D (x-y)^2 \, dA$$

LET $\quad G(u,v) = (x(u,v), y(u,v))$

WHERE $\quad x(u,v) = \frac{1}{2}(u+v)$

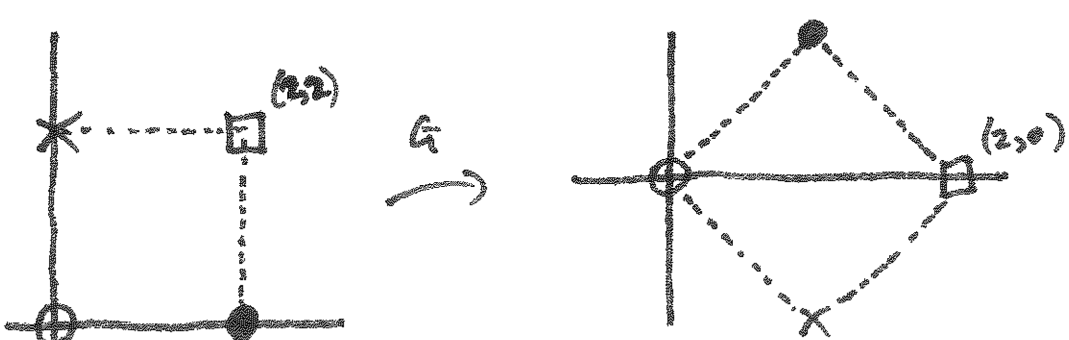
$\qquad\qquad y(u,v) = \frac{1}{2}(u-v)$

(WE'LL COME BACK TO HOW WE THOUGHT OF THIS)

NOTICE $\quad J(G) = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$

SO $\quad \det J(G) = \frac{1}{2}\left(\frac{-1}{2}\right) - \frac{1}{2}\cdot\frac{1}{2} = \frac{-1}{2}.$

(THIS IS NEVER 0 SO G IS A COORDINATE CHANGE)



G TAKES $[0,2] \times [0,2]$ TO D.

$$\iint_D (x-y)^2 \, dA = \int_0^2 \int_0^2 v^2 \cdot \frac{1}{2} \, du \, dv$$

$$= \left[\frac{v^3}{3} \right]_0^2 = \frac{8}{3}.$$

WHY G?

D CAN BE DESCRIBED AS

$$0 \leq x+y \leq 2$$
$$0 \leq x-y \leq 2.$$

SO WANT $\quad u = x+y$

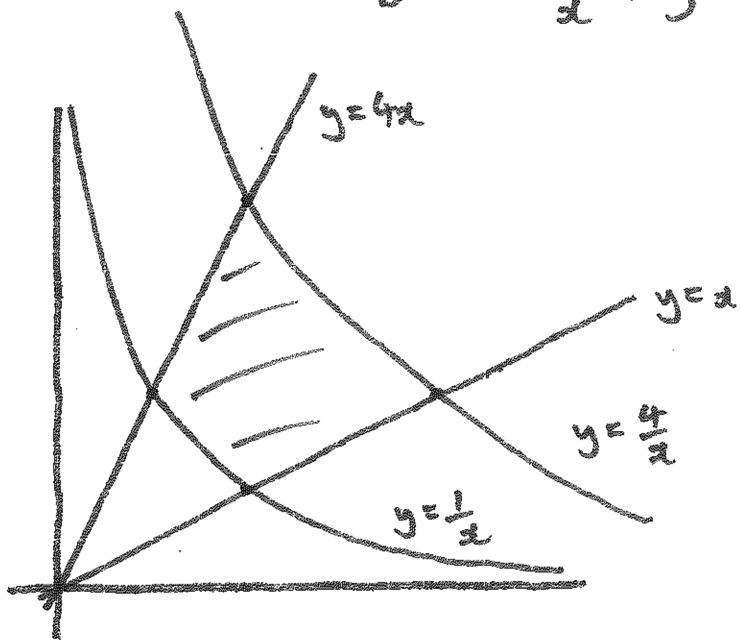
$\qquad\qquad\qquad v = x-y$

$$\rightsquigarrow \quad x = \frac{1}{2}(u+v)$$
$$y = \frac{1}{2}(u-v).$$

EXAMPLE

$$\iint_D x^2 + y^2 \, dA \quad \text{WHERE}$$

$$D : \ \frac{1}{x} \le y \le \frac{4}{x}, \quad x \le y \le 4x$$



REARRANGE EQV. $\quad x \geq 0, \quad 1 \le xy \le 4, \quad 1 \le \frac{y}{x} \le 4.$

$$\begin{aligned} u &= xy \\ v &= \frac{y}{x} \end{aligned} \ ? \Leftarrow \begin{aligned} \frac{u}{v} &= x^2 \\ uv &= y^2 \end{aligned}$$

ANNOYING SQUARE ROOTS.

$$\begin{aligned} u^2 &= xy \\ v^2 &= \frac{y}{x} \end{aligned} \Leftarrow \begin{aligned} \frac{u^2}{v^2} &= x^2 \\ u^2v^2 &= y^2 \end{aligned} \Leftarrow \boxed{\begin{aligned} \frac{u}{v} &= x \\ uv &= y \end{aligned}}$$

LET $G(u,v) = (x(u,v), y(u,v))$ WHERE $\begin{aligned} x(u,v) &= \frac{u}{v} \\ y(u,v) &= uv. \end{aligned}$

WE SEE THAT $G\left([1,2]\times[1,2]\right) = D$.

$\det J(G) = 2\frac{u}{v}$

$$\iint_D x^2+y^2 \, dA = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right) \frac{2u}{v} \, du \, dv$$

$$= 2 \int_1^2 u^3 \, du \int_1^2 \frac{1}{v^3} + \frac{1}{v} \, dv .$$

EXAMPLE

$$\iint_D xy\left(x^2+y^2\right)\,dA$$

$$D: \quad -3 \le x^2-y^2 \le 3, \quad 1 \le xy \le 4$$

$$F(x,y) = \left(x^2-y^2,\, xy\right) \qquad G = F^{-1}.$$

$$\mathrm{JAC}(F) = 2\left(x^2+y^2\right)$$

$$\iint_D xy\left(x^2+y^2\right)\,dA = \int_{-3}^{3}\int_{1}^{4} \frac{v}{2}\,dv\,du = \frac{45}{2}.$$