PATH-INDEPENDENCE REVISITED

AND SURFACE INDEPENDENCE

RECALL: IF C IS AN ORIENTED CURVE FROM P TO Q,

AND f IS DIFFERENTIABLE NEAR C,

THEN
$$\int_C \nabla f \cdot d\underline{r} = f(Q) - f(P).$$

LET $\underline{F} = \nabla f$.

CONSEQUENCES: 1) CIRCULATIONS OF $\underline{F}$ ARE 0.

2) VECTOR LINE INTEGRALS OF $\underline{F}$ ONLY DEPEND ON THE END POINTS OF C.

ALSO, $\nabla \times \underline{F} = \underline{0}$.

WE'VE SEEN THAT THE CONDITION $\nabla \times \underline{F} = 0$ DOES NOT ALWAYS GUARANTEE THE EXISTENCE OF A POTENTIAL f.

RECALL: IF S IS AN ORIENTED SURFACE W/ BOUNDARY ∂S GIVEN THE INDUCED ORIENTATION,

AND $\underline{A}$ IS DIFFERENTIABLE NEAR S,

THEN

$$\iint_S (\nabla \times \underline{A}) \cdot d\underline{S} = \int_{\partial S} \underline{A} \cdot d\underline{r}.$$

LET $\underline{F} = \nabla \times \underline{A}$.

$\underline{A}$ IS CALLED A <u>VECTOR POTENTIAL</u> FOR $\underline{F}$

CONSEQUENCES:

1) FLUX INTEGRALS OF $\underline{F}$ ON CLOSED SURFACES ARE 0.

(A SURFACE S IS CLOSED IF $\partial S = \phi$ = NOTHING, SO $\int_{\partial S}$ ANYTHING $= 0$.)

2) FLUX INTEGRALS OF $\underline{F}$ ONLY DEPEND ON THE BOUNDARY OF THE SURFACE.

WE SAY $\underline{F}$ IS SURFACE INDEPENDENT.

ALSO, NOTICE $\nabla \cdot \underline{F} = 0$.

$\nabla \cdot \underline{F} = 0$ DOES NOT ALWAYS GUARANTEE THE EXISTENCE OF A VECTOR POTENTIAL $\underline{A}$: WE'LL SAY MORE ABOUT THIS.

EXAMPLE : LET $\underline{A} = (y^2, x, 0)$.

LET S BE THE SURFACE

$$\left(\sqrt{x^2+z^2} - 2\right)^2 + y^2 = 1, \quad z \geq 0$$

WITH OUTWARD POINTING NORMAL.

CALCULATE THE FLUX

$$\iint_S (\nabla \times \underline{A}) \cdot d\underline{S} .$$

$\nabla \times \underline{A} = (0, 0, 1-2y)$.

BY SURFACE INDEPENDENCE

$$\iint_S (\nabla \times \underline{A}) \cdot d\underline{S} = \iint_{\substack{(x+2)^2+y^2=1 \\ \circlearrowleft \ z=0}} (\nabla \times \underline{A}) \cdot d\underline{S} + \iint_{\substack{(x-2)^2+y^2=1 \\ \circlearrowright \ z=0}} (\nabla \times \underline{A}) \cdot d\underline{S}$$

$$= \iint_{\cdots} 1-2y \, dS + \iint_{\cdots} 1-2y \, dS$$

BY SYMM. CAN IGNORE y STUFF. $\longrightarrow$

$$= 2 \text{ AREA}(\text{UNIT DISK}) = 2\pi.$$

$\nabla \times \underline{F} = \underline{0}$ IS A WEAKER CONDITION THAN $\underline{F} = \nabla f$.

WHAT DOES IT GIVE US? LOCAL PATH-INDEPENDENCE

USE STOKES' AGAIN:

IF S IS AN ORIENTED SURFACE
W/ BOUNDARY ∂S GIVEN THE INDUCED ORIENTATION,

AND $\underline{F}$ IS DIFFERENTIABLE NEAR S WITH $\nabla \times \underline{F} = \underline{0}$,

THEN
$$\int_{\partial S} \underline{F} \cdot d\underline{r} = 0.$$

(BECAUSE $\displaystyle\int_{\partial S} \underline{F} \cdot d\underline{r} = \iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = \iint_S \underline{0} \cdot d\underline{S} = 0.$)

CONSEQUENCES OF $\nabla \times \underline{F} = \underline{0}$:

1) CIRCULATIONS OF $\underline{F}$ ARE 0

IF THE CLOSED CURVE IS THE BOUNDARY OF A SURFACE ON WHICH $\underline{F}$ IS DEFINED AND DIFFERENTIABLE.

2) $\displaystyle\int_{C_1} \underline{F} \cdot d\underline{r} = \int_{C_2} \underline{F} \cdot d\underline{r}$ IF $(-C_1) \cup C_2$ IS THE BOUNDARY OF A SURFACE ON WHICH $\underline{F}$ IS DEFINED AND DIFF.

EXAMPLES : SEE LAST TIME.

LET $\underline{V}$ BE VORTEX VECTOR FIELD.

1)
$$\int_{\substack{(x+\frac{1}{2})^6 + (x+\frac{1}{2}) + y^6 = 0 \\ \circlearrowleft \quad z = 0}} \underline{V} \cdot d\underline{r} = 0.$$

2)
$$\int_{\substack{(x-\frac{1}{2})^6 + (x-\frac{1}{2}) + y^6 = 0 \\ \circlearrowleft \quad z = 0}} \underline{V} \cdot d\underline{r}$$

$$= \int_{\substack{x^2+y^2 = 1 \\ \circlearrowleft \; z=0}} \underline{V} \cdot d\underline{r} = 2\pi.$$

$\nabla \cdot \underline{F} = 0$ IS A <u>WEAKER</u> CONDITION THAN $\underline{F} = \nabla \times \underline{A}$.

WHAT DOES IT GIVE US? <u>LOCAL</u> SURFACE-INDEPENDENCE

USE DIVERGENCE THM.

IF E IS A 3D REGION
W/ BOUNDARY ∂E GIVEN OUTWARD POINTING NORMAL,

AND $\underline{F}$ IS DIFFERENTIABLE NEAR E WITH $\nabla \cdot \underline{F} = 0$,

THEN
$$\iint_{\partial E} \underline{F} \cdot d\underline{S} = 0.$$

(BECAUSE $\displaystyle\iint_{\partial E} \underline{F} \cdot d\underline{S} = \iiint_E \nabla \cdot \underline{F} \, dV = \iiint_E 0 \, dV = 0.$)

CONSERVENCES OF $\nabla \cdot \underline{F} = 0$:

1) FLUX INTEGRALS OF $\underline{F}$ ON CLOSED SURFACES ARE 0
<u>IF</u> THE CLOSED SURFACE IS THE BOUNDARY OF A REGION ON WHICH $\underline{F}$ IS <u>DEFINED</u> AND DIFF.

2) $\displaystyle\iint_{S_1} \underline{F} \cdot d\underline{S} = \iint_{S_2} \underline{F} \cdot d\underline{S}$ <u>IF</u> $(-S_1) \cup S_2$ IS THE BOUNDARY OF A REGION ON WHICH $\underline{F}$ IS <u>DEFINED</u> AND DIFFERENTIABLE.

EXAMPLES :

$$\underline{F} = \frac{(x,y,z)}{\rho^3}$$

IS THE ANALOG OF THE VORTEX VECTOR FIELD IN THE NEXT DIMENSION UP.

$$\nabla \cdot \underline{F} = 0$$

BUT YOU CANNOT FIND A VECTOR POTENTIAL FOR $\underline{F}$ DEFINED ON ALL OF $\mathbb{R}^3 - (0,0,0)$.

(YOU CAN IF YOU REMOVE AN ENTIRE AXIS.

$$\nabla \times \frac{(yz, -zx, 0)}{\rho(x^2+y^2)} = \underline{F}$$

DEFINED ON $\mathbb{R}^3 - (z\text{-AXIS})$.)

THIS IS BECAUSE

$$\iint_{\substack{x^2+y^2+z^2=1 \\ \text{OUTWARD} \\ \text{NORMAL}}} \underline{F} \cdot d\underline{S} = 4\pi \neq 0.$$

1) $$\iint_{\substack{(x-2)^2+y^2+z^2=1 \\ \text{OUTWARD} \\ \text{NORMAL}}} \underline{F} \cdot d\underline{S} = 0$$

SINCE $\underline{F}$ IS DEFINED THROUGHOUT

$$(x-2)^2 + y^2 + z^2 \leq 1.$$

2) $$\iint_{\substack{x^2+y^2+z^2=2 \\ \text{OUTWARD} \\ \text{NORMAL}}} \underline{F} \cdot d\underline{S} = 4\pi$$

SINCE $\underline{F}$ IS DEFINED THROUGHOUT

$$1 \leq x^2 + y^2 + z^2 \leq 2.$$