1-VARIABLE CALCULUS QUESTION:

WHAT IS SIGNED AREA UNDER THE CURVE?

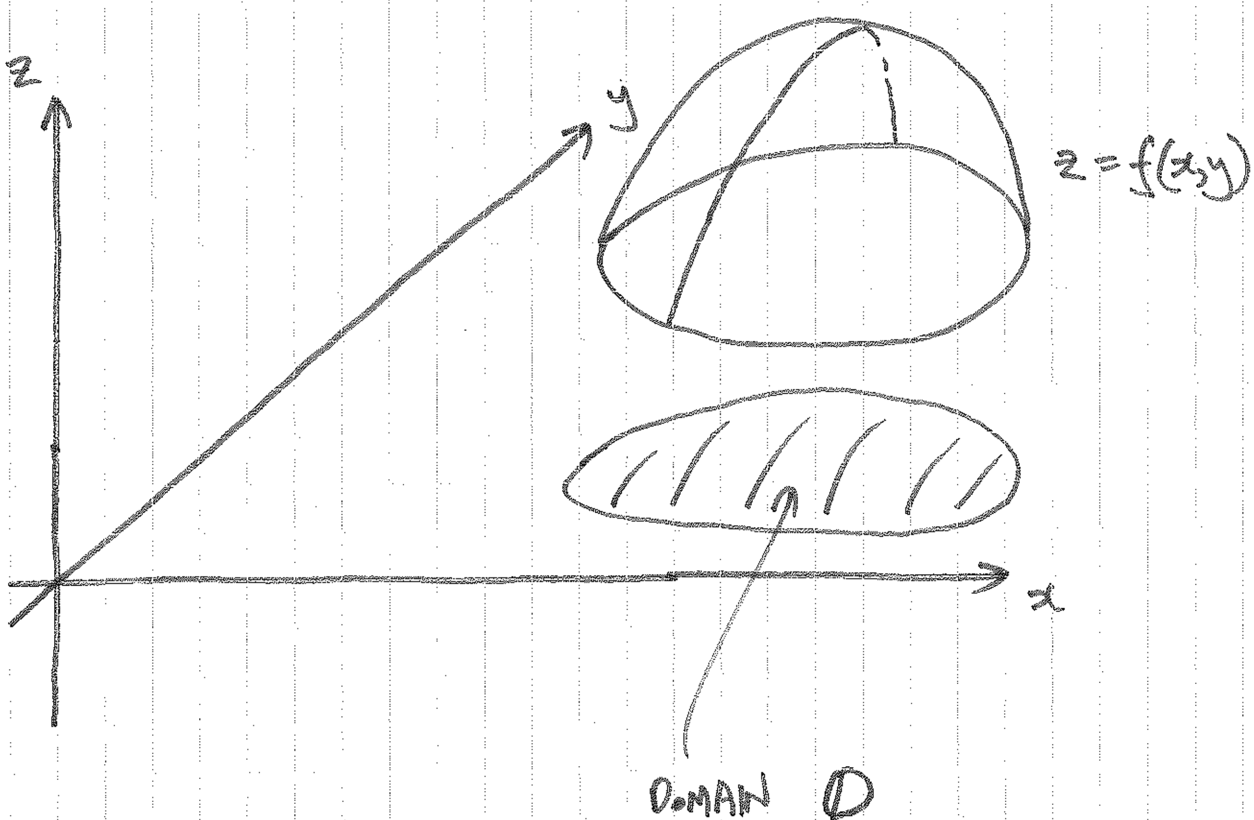
ANSWER: $\int_a^b f(x)\,dx$

DEFINED AS A LIMIT OF AREAS OF LOT OF RECTANGLES.

FTC ALLOWS YOU TO CALCULATE MORE EFFICIENTLY.

2-VARIABLE CALCULUS QUESTION

SUPPOSE HAVE A SURFACE



WHAT IS SIGNED VOLUME UNDER SURFACE ?

ALTERATIVELY, IF $f(x,y)$ REPRESENTS "DENSITY",
WHAT IS "WEIGHT" OF $\mathbb{D}$?

(" " USED BECAUSE MORE CONFUSING WHEN $f(x,y) < 0$).

ANSWER :

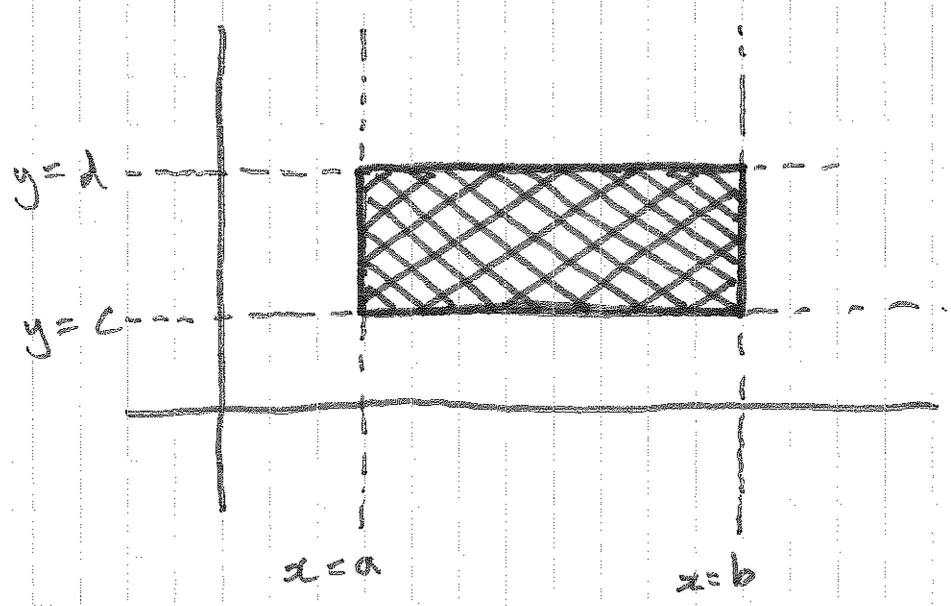
$$\iint_{\mathbb{D}} f(x,y)\, dA$$

HAS A DEFINITION IN TERMS OF VOLUME OF LOTS OF CUBOIDS.

D COULD BE CRAZY.

FOR NOW, LET'S NOT GO CRAZY (SORRY PRINCE):

CONSIDER $D = R = [a,b] \times [c,d]$ A RECTANGLE



THM: IF $f(x,y)$ IS CONTINUOUS ON $R = [a,b] \times [c,d]$,

THEN WE CAN INTEGRATE $f(x,y)$ AND

$$\iint_R f(x,y)\, dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\, dy\, dx$$

$$= \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\, dx\, dy.$$

SOME Qs :

1) WHAT DO THE EXPRESSIONS IN THIS THM MEAN ?

2) WHAT'S THE POINT OF THIS THM ?

3) DOES THIS THEOREM AGREE WITH OUR INTUITION ?

ANSWERS

1) a) $\int_R f(x,y)\,dA$ HAS A DEFN WHICH WE'RE NEGLECTING TO GIVE.
IT IS THE CORRECT DEFN, [redacted] DEFINES "SIGNED VOLUME UNDER SURFACE", BUT WE'LL NEVER USE IT.

b) $\int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx$ MEANS

$$\int_{x=a}^{b} \left(\int_{y=c}^{d} f(x,y)\,dy \right) dx$$

HOLD x CONSTANT AND DO A REGULAR 1-D INTEGRAL. WRT y.

THEN DO AN INTEGRAL WRT x.

c) SIMILAR.

2) SOLVES A NEW PROBLEM IN TERMS OF THINGS THAT WE ALREADY UNDERSTAND

3) ...

BEFORE THINKING ABOUT 3) FURTHER LET'S SEE EXAMPLES.

EXAMPLE

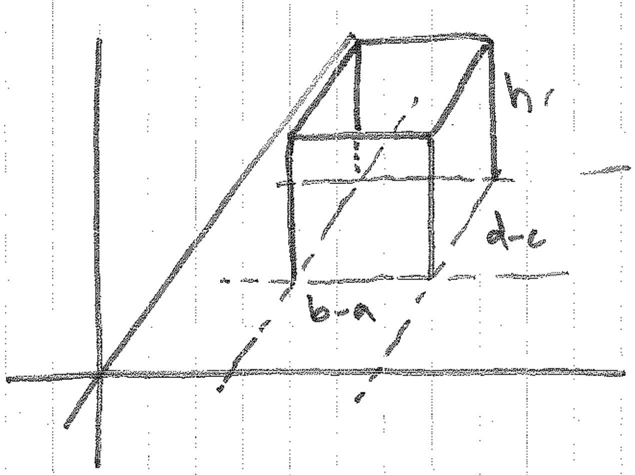
I) LET $R = [a,b] \times [c,d]$, $f(x,y) = h$.

THEN

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d h \, dy \, dx$$

$$= \int_a^b h(d-c)\, dx = h(d-c)(b-a)$$

$$= (b-a)(d-c)\, h$$



✓

II) ...

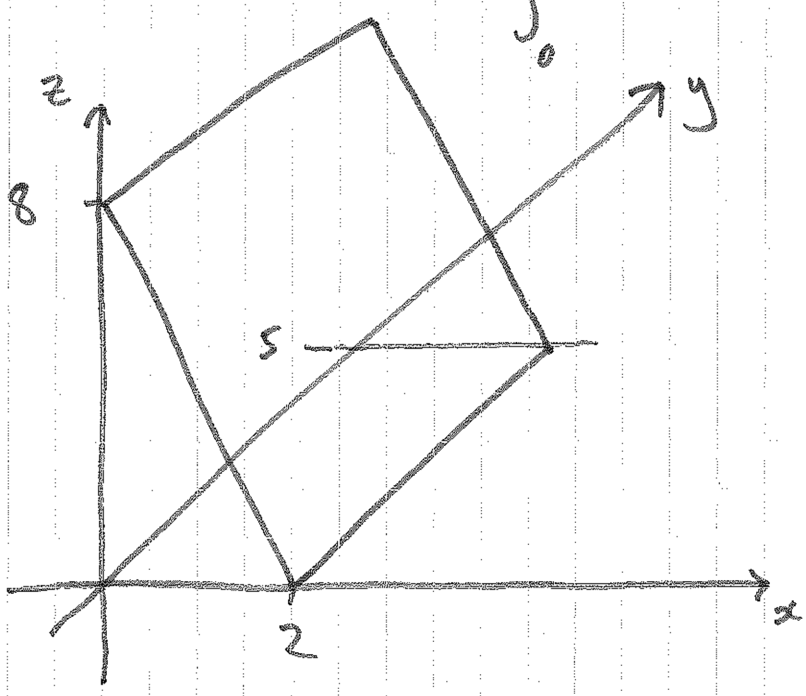
II) LET $R = [0,2] \times [0,5]$, $f(x,y) = 8 - 4x$

THEN

$$\iint_R f(x,y)\, dA = \int_0^5 \int_0^2 8 - 4x \, dx\, dy$$

$$= \int_0^5 \left[8x - 2x^2 \right]_0^2 dy$$

$$= \int_0^5 8\, dy = 40.$$



TRIANGULAR PRISM, so $\frac{1}{2} \cdot 2 \cdot 8 \cdot 5 = 40$ ✓.

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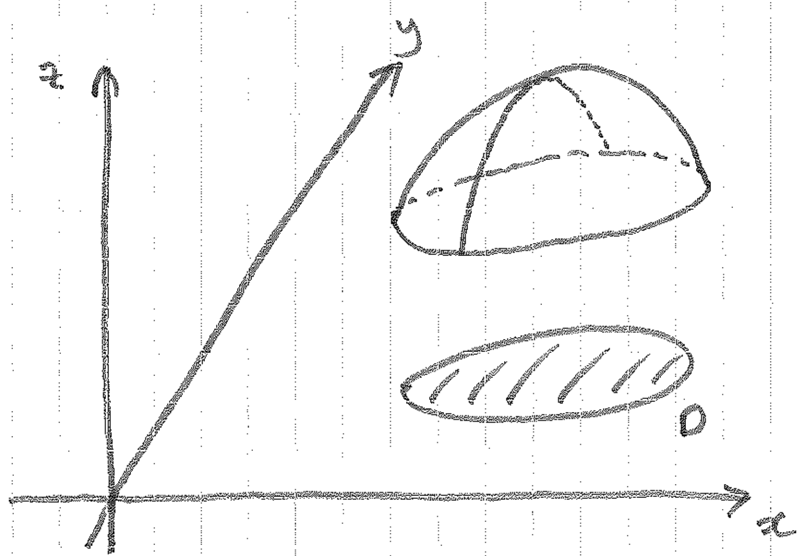
LAST TIME

$$\iint_D f(x,y)\,dA$$

IS SIGNED VOLUME UNDER SURFACE $z = f(x,y)$



IF $f(x,y)$ IS CONT. ON $R = [a,b] \times [c,d]$, CAN INTEGRATE f AND

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx = \dots$$

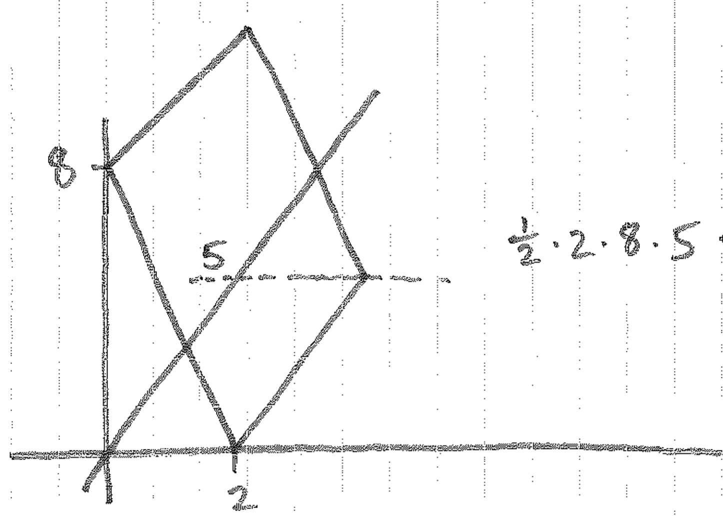
NEW

TWO 1-D INTEGRALS

EXAMPLE II $\quad R = [0,2] \times [0,5]$, $\quad f(x,y) = 8 - 4x$

SHOWED $\iint_R f(x,y)\,dA = 40.$

PICTURE:



$\frac{1}{2} \cdot 2 \cdot 8 \cdot 5 = 40.$

BACK TO 3) : DOES THE THM AGREE WITH OUR INTUITION?

TWO EXAMPLES CHECKED OUT.

ALSO, ...

SUPPOSE HAVE $R = [a,b] \times [c,d]$ AND $f(x,y)$.

LET

$$S(x) = \int_c^d f(x,y)\, dy \qquad \text{A FUNCTION OF } x.$$

THM SAYS

$$\iint_R f(x,y)\, dA = \int_a^b S(x)\, dx.$$

WHAT IS S?

$S(x_0)$ = AREA OF CROSS-SECTION IN VERTICAL PLANE $x = x_0$ PERPENDICULAR TO x-AXIS.

VOLUME = INTEGRAL OF CROSS-SECTIONAL AREAS.

SEE TEXTBOOK FOR NICE PICTURES.

TWO MORE EXAMPLES (USING SYMMETRY)

1) $R = [-3,3] \times [0,1]$

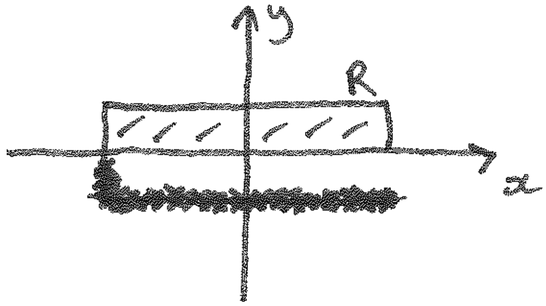
$$\iint x e^{-y^2} \, dA.$$

a) $dy\,dx$: IMPOSSIBLE

b) $dx\,dy$: CALCULATE TO GET 0.

c) SAY IS 0 BECAUSE OF SYMMETRY

- THE DOMAIN R IS SYMMETRIC ABOUT $x = 0$



- THE GRAPH OF $z = xe^{-y^2}$ IS ANTI-SYMMETRIC ABOUT THE PLANE $x = 0$.

  IF YOU HAVE A MAC, CLICK Q, TYPE "GRAPHER", 3D, $z = xe^{-y^2}$ TO SEE THIS.

2) $R = [-1,1] \times [-1,1]$

$$\iint_R (x+y)(x^2+y^2) + 1 \, dA = 0 + 0 + 4$$

- DOMAIN IS SYMMETRIC ABOUT $x = 0$ AND $y = 0$.
- $x(x^2+y^2)$ IS ANTI-SYMMETRIC ABOUT $x = 0$; $y(x^2+y^2)$ ... ABOUT $y = 0$.

SUPPOSE D IS A CLOSED DOMAIN

CONTAINS ITS BOUNDARY ∂D

SUPPOSE ∂D IS SIMPLE CLOSED CURVE, PIECEWISE SMOOTH

DOES NOT CROSS ITSELF

MAKES A LOOP

CAN JOIN

(UNFORTUNATE "CLOSED" MEANS DIFFERENT THINGS)

SUPPOSE $f(x,y)$ IS DEFINED ON D.

$$\iint_D f(x,y)\,dA := \iint_R \tilde{f}(x,y)\,dA$$

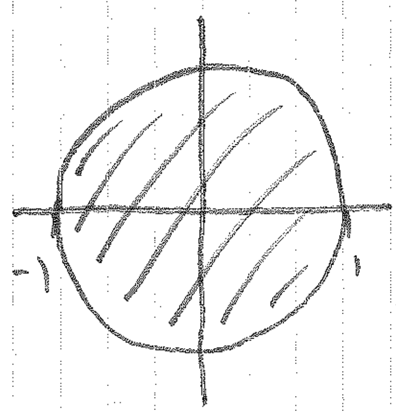
WHERE R IS A RECTANGLE CONTAINING D,

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{IF } (x,y) \text{ IS IN } D \\ 0 & \text{IF } (x,y) \text{ IS NOT IN } D. \end{cases}$$

NOTE $\text{AREA}(D) = \displaystyle\iint_D 1\,dA.$

HOW TO CALCULATE ?

EXAMPLE : LET $D = \{(x,y) : x^2+y^2 \le 1\}$



$$\iint_D 1 \, dA \ ?$$

ANSWER SHOULD BE π.

- $\tilde{f}(x,y) = \begin{cases} 1 & x^2+y^2 \le 1 \\ 0 & x^2+y^2 > 1. \end{cases}$

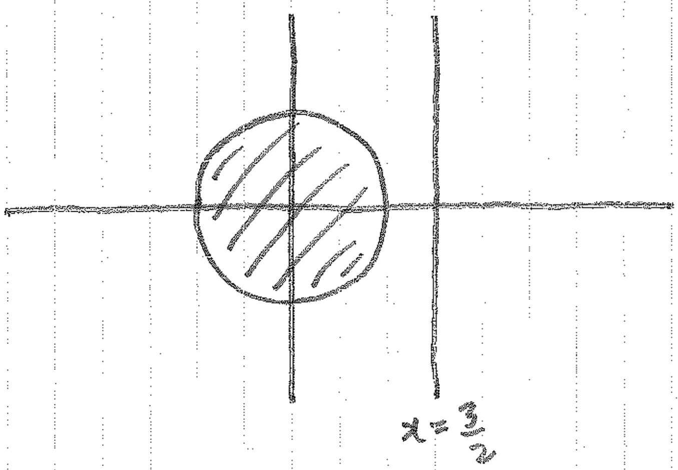
- $R = [-2,2] \times [-2,2]$ CONTAINS D.

  SO BY DEFN

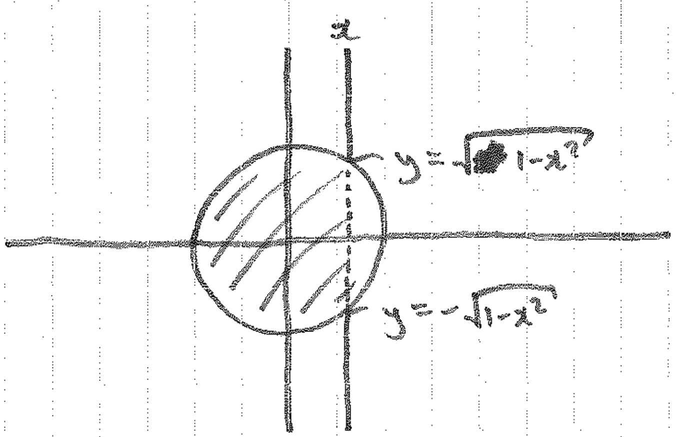
- $$\iint_D 1 \, dA = \iint_R \tilde{f}(x,y) \, dA = \int_{-2}^{2}\int_{-2}^{2} \tilde{f}(x,y) \, dy \, dx.$$

LET'S THINK ABOUT

$$S(x) = \int_{-2}^{2} \tilde{f}(x,y)\, dy .$$



WHEN $|x| > 1$, $x^2 + y^2 > 1$, SO $\tilde{f}(x,y) = 0$, SO $S(x) = 0$.



FIX x WITH $|x| \leq 1$.

$\tilde{f}(x,y)$ IS DISCONTINUOUS IN y.

$$\tilde{f}(x,y) = \begin{cases} 1 & -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \\ 0 & y < -\sqrt{1-x^2} \text{ OR } y > \sqrt{1-x^2} \end{cases}$$

SO WHEN $|x| \leq 1$,

$$S(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dA$$

LIMITS BECOME FUNCTIONS OF x.

$$\iint_D 1 \, dA = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dy \, dx$$

MOST EXTREME VALUES OF x IN D.

HAVING FIXED AN x, MOST EXTREME VALUES OF y SUCH THAT (x,y) IS IN D.

$$= \int_{-1}^{1} 2\sqrt{1-x^2} \, dx$$

$$= \left[x\sqrt{1-x^2} + \text{ARCSIN}(x) \right]_{-1}^{1} = \pi.$$

SAY A DOMAIN D IS <u>VERTICALLY SIMPLE</u> IF

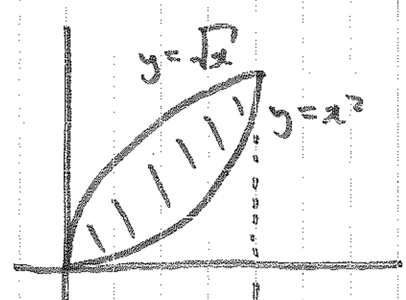
$$D = \{ (x,y) : a \le x \le b, \ g_1(x) \le y \le g_2(x) \}$$

IN THIS CASE,

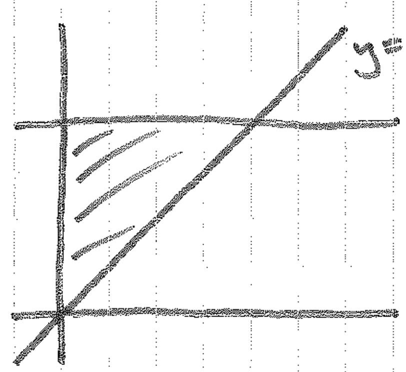
$$\iint_D f(x,y)\, dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx .$$

... <u>HORIZONTALLY SIMPLE</u>

EXAMPLES : $\displaystyle\iint_D x^2 y \, dA$



$\displaystyle\iint_D e^{y^2} \, dA$



DO dy, dx
THEN dx, dy .