Math 32B
Calculus of Several Variables

Final

Instructions: You have 180 minutes to complete this exam. There are eight questions, worth a total of 88 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ________________________

Discussion: ____________________________

Question	Points	Score
1	14	
2	10	
3	8	
4	18	
5	12	
6	10	
7	8	
8	8	
Total:	88	

Problem 1.

(a) [7pts.] Let C be the half of the unit circle $x^2 + z^2 = 1$, $y = 0$, with $z \leq 0$. Orient C in the clockwise direction when viewed from the negative y-axis. Let $\mathbf{F}(x, y, z) = (y^4, z^3, 1 - x^2)$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

> **Solution:** $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (0, 0, 1 - x^2) \cdot d\mathbf{r} = 0$.

(b) [7pts.] Let C be the curve in the xy-plane described by $z = 0$, $y = x^2$, $-2 \leq x \leq 2$. Let $f(x, y, z) = 12(x + \sqrt{y} + z^3)$. Calculate $\int_C f\ ds$.

> **Solution:** $\int_C f\ ds = \int_C 12\sqrt{y}\ ds = 2\int_0^2 12t\sqrt{1 + 4t^2}\ dt = 2(17^{\frac{3}{2}} - 1)$.

Problem 2.

Let S be the part of the cone $x^2 + y^2 = 4z^2$ between the planes $z = 2$ and $z = 5$, and in the second xy-quadrant $x \leq 0$, $y \geq 0$.

(a) [3pts.] Which two of the following vectors are tangent vectors to S at $(-3, 4, \frac{5}{2})$?

$$(0, 5, 2),\ (0, 5, -2),\ (-2, 1, 1),\ (-3, 4, 0),\ (3, 4, 0),\ (-4, 3, 0).$$

> **Solution:** The tangent plane at $(-3, 4, \frac{5}{2})$ has normal $(-3, 4, -10)$.
> So $(0, 5, 2)$ and $(-2, 1, 1)$ are tangent vectors.

(b) [7pts.] Let $f(x, y, z) = x + z$. Show that

$$\iint_S f\ dS = 39\sqrt{5}(\pi - 4).$$

> **Solution:** Let $G(\theta, z) = (2z\cos\theta, 2z\cos\theta, z)$. Then $\|G_\theta \times G_z\| = 2\sqrt{5}z$.
> $2\sqrt{5}\ \int_{\theta=\frac{\pi}{2}}^{\pi} (2\cos\theta + 1)\ d\theta\ \int_{z=2}^{5} z^2\ dz = 2\sqrt{5} \cdot (\frac{\pi}{2} - 2) \cdot \frac{5^3 - 2^3}{3} = 39\sqrt{5}(\pi - 4)$.

Problem 3. *8pts.*

Let $\mathcal{E}$ be the region enclosed by the surfaces $x^2 + y^2 + z^2 = 8$ and $x^2 + y^2 = 2z$ in the octant $x, y, z \geq 0$.

Let $f(x, y, z) = 3z$. Calculate $\iiint_{\mathcal{E}} f\ dV$.

> **Solution:** $\frac{\pi}{2}\int_{r=0}^{2}\int_{z=\frac{r^2}{2}}^{\sqrt{8-r^2}} 3z\ dz\ r\ dr = \frac{\pi}{4}\int_{r=0}^{2}(24r - 3r^3 - \frac{3r^5}{4})\ dr = 7\pi$.

Problem 4.

In each of the parts (a)-(d), you should justify your answer fully.

(a) [4pts.] Is $\mathbf{F}(x, y, z) = \frac{(x,y,0)}{x^2+y^2}$ conservative on $\mathbb{R}^3 - (z\text{-axis})$?

(b) [4pts.] Is $\mathbf{F}(x, y, z) = \frac{(-y,x,0)}{x^2+y^2}$ conservative on $\mathbb{R}^3 - (z\text{-axis})$?

(c) [4pts.] Does $\mathbf{F}(x, y, z) = (x, y, z)$ have a vector potential on $\mathbb{R}^3 - \{(0, 0, 0)\}$?

(d) [4pts.] Does $\mathbf{F}(x, y, z) = (0, 0, 1)$ have a vector potential on $\mathbb{R}^3 - \{(0, 0, 0)\}$?

(e) [2pts.] Write down a domain on which there exist well-defined differentiable vector fields $\mathbf{F}$ and $\mathbf{G}$ such that $\nabla \times \mathbf{F} = \mathbf{0}$, $\nabla \cdot \mathbf{G} = 0$, $\mathbf{F}$ is not conservative, and $\mathbf{G}$ does not have a vector potential.

Solution:

(a) Yes, $\frac{1}{2}\ln(x^2 + y^2)$ is a potential.

(b) No, since $\int_{z=0,\ x^2+y^2=1,\ \text{counter-clockwise}} \mathbf{F} \cdot d\mathbf{r} = 2\pi \neq 0$.

(c) No, since $\nabla \cdot \mathbf{F} = 3 \neq 0$.

(d) Yes, $(0, x, 0)$ is a vector potential.

(e) $\mathbb{R}^3 - z\text{-axis} - \{(1, 0, 0)\}$.

Problem 5.

Let $\mathcal{E}$ be the solid half-cylinder $x^2 + y^2 \leq 1$, $x \geq 0$, $0 \leq z \leq 1$.

Let S be the surface $x^2 + y^2 = 1$, $x \geq 0$, $0 \leq z \leq 1$.

Give S the unit normal pointing in the positive x-direction.

For example, at $(1, 0, \frac{1}{2})$, a normal pointing in the correct direction is given by $(1, 0, 0)$.

Let $\mathbf{F}(x, y, z) = (y^2 z, 4y, 1)$.

(a) [4pts.] Calculate the volume integral

$$\iiint_{\mathcal{E}} (\nabla \cdot \mathbf{F})\ dV.$$

(b) [8pts.] Calculate the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Solution:

(a) $\iiint_{\mathcal{E}} (\nabla \cdot \mathbf{F})\ dV = \iiint_{\mathcal{E}} 4\ dV = 4 \cdot \mathbf{Vol}(\mathcal{E}) = 2\pi$.

(b) $\iint_{\partial\mathcal{E}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{E}} (\nabla \cdot \mathbf{F})\ dV = 2\pi$.

$\iint_{\text{top} \,\cup\, \text{bottom}} \mathbf{F} \cdot d\mathbf{S} = 0$ since $\mathbf{F} \cdot \mathbf{n} = 1$ on the top and $\mathbf{F} \cdot \mathbf{n} = -1$ on the bottom.

Thus, $\iint_S \mathbf{F} \cdot d\mathbf{S} = 2\pi + \int_{y=-1}^{1} y^2\ dy \int_{z=0}^{1} z\ dz = 2\pi + \frac{1}{3}$.

Problem 6. *10pts.*

Let S be the hyperboloid $x^2 + y^2 = z^2 + 1$, $1 \leq z \leq 3$, with downward pointing normal.

For example, at $(2, 1, 2)$, a normal pointing in the correct direction is given by $(2, 1, -2)$.

Let $\mathbf{A}(x, y, z) = (y^2, xz, 0)$ and $\mathbf{F} = \nabla \times \mathbf{A} = (-x, 0, z - 2y)$.

Calculate the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Solution: By surface-independence,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{z=1,\ x^2+y^2 \leq 2} (z - 2y)\ dS - \iint_{z=3,\ x^2+y^2 \leq 10} (z - 2y)\ dS \\
&= \iint_{x^2+y^2 \leq 2} 1\ dA - \iint_{x^2+y^2 \leq 10} 3\ dA = 1 \cdot \pi \cdot 2 - 3 \cdot \pi \cdot 10 = -28\pi.
\end{aligned}$$

Problem 7.

Let C be the oriented curve parametrized by

$$\mathbf{r}(t) = (2,0,2) + \cos t(-1,1,1) + \sin t(-1,-2,1),\ \ 0 \leq t \leq 2\pi.$$

This parametrization is fairly nasty, so to help you...

C is an ellipse on the plane $x+z=4$ which encloses an area of $3\pi\sqrt{2}$. The orientation is counter-clockwise when one looks down on it, that is, if you look at it from $(8,0,8)$.

Let $\mathbf{F}(x,y,z) = (e^{x^2}, x + xy + yz, y)$.

(a) [2pts.] State Stoke's theorem.

> **Solution:** Omitted.

(b) [6pts.] Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

> **Solution:** By Stokes' theorem, it's equal to
>
> $$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} &= \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS \\ &= \iint_S (1-y,0,1+y)\cdot\frac{(1,0,1)}{\sqrt{2}}\ dS = \iint_S \sqrt{2}\ dS = 6\pi.\end{aligned}$$

Problem 8.

Let S_1 be the upper hemisphere of the unit sphere: $x^2 + y^2 + z^2 = 1,\ z \geq 0$.

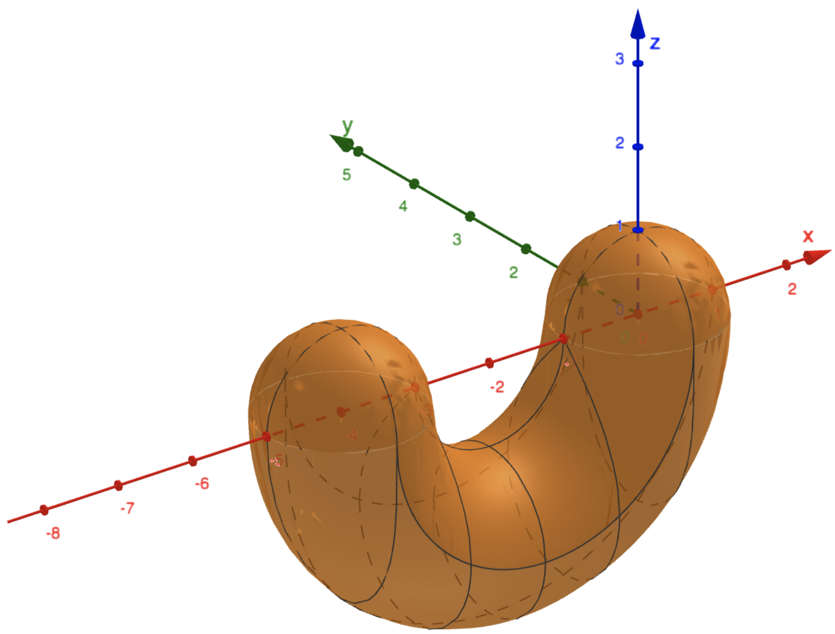
Let S_2 be S_1 shifted to the left by 4: $(x+4)^2 + y^2 + z^2 = 1,\ z \geq 0$.

Let S_3 be the lower half of a torus (donut):

$$\left(\sqrt{(x+2)^2 + z^2} - 2\right)^2 + y^2 = 1,\ z \leq 0,$$

which meets the hemispheres.

$S = S_1 \cup S_2 \cup S_3$ is a closed surface. Give it an outward pointing normal.



Let $\mathbf{F}(x,y,z) = \frac{2(x,y,z)}{\rho^3} = \frac{2(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$. It is true that $\nabla \cdot \mathbf{F} = 0$.

(a) [3pts.] State the divergence theorem carefully.

> **Solution:** Omitted.

(b) [5pts.] Use the divergence theorem to calculate the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

> **Solution:** Let S_4 be the sphere $x^2+y^2+z^2 = \frac{1}{100}$ with outward pointing normal. Then $(-S_4) \cup S$ is $\partial\mathcal{E}$ where $\mathcal{E}$ is the region inside S and outside S_4. Moreover, $\mathbf{F}$ is defined and differentiable on $\mathcal{E}$. We have $\iiint_{\mathcal{E}} (\nabla \cdot \mathbf{F})\, dV = 0$, so the divergence theorem gives $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 8\pi$.