Math 32B: Conservative and not conservative fields, $d\theta_1 - d\theta_2$

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Remark: the vector fields in this assignment have the form

$$\mathbf{F}(x, y, z) = (F_1(x, y), F_2(x, y), 0).$$

This means that they can be visualized entirely in 2D and that their curls are multiples of $(0, 0, 1)$. For questions 1-5, the 2D projections of the domains are $\mathbb{R}^2 - (0, 0)$, $\mathbb{R}^2 - (0, 0)$, $\mathbb{R}^2 - (0, 0)$, $\mathbb{R}^2-$(the line from $(0, -1)$ to $(0, 1)$), and $\mathbb{R}^2 - (0, -1) - (0, 1)$, respectively.

1. Consider the following vector field with domain $\mathbb{R}^3-$(z-axis):

$$\mathbf{F}(x, y, z) = \frac{(-y, x, 0)}{\sqrt{x^2 + y^2}}.$$

   (a) What is $\nabla \times \mathbf{F}$?
   (b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ where $0 \leq t \leq 2\pi$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
   (c) Is $\mathbf{F}$ conservative on $\mathbb{R}^3-$(z-axis)?
   (d) How would you justify your answer to (c) as quickly as possible?

2. Consider the following vector field with domain $\mathbb{R}^3-$(z-axis):

$$\mathbf{F}(x, y, z) = \frac{(-y, x, 0)}{x^2 + y^2}.$$

   (a) What is $\nabla \times \mathbf{F}$?
   (b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ where $0 \leq t \leq 2\pi$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
   (c) Is $\mathbf{F}$ conservative on $\mathbb{R}^3-$(z-axis)?
   (d) How would you justify your answer to (c) as quickly as possible?

3. Consider the following vector field with domain $\mathbb{R}^3 - (z\text{-axis})$:

$$\mathbf{F}(x, y, z) = \frac{(x, y, 0)}{\sqrt{x^2 + y^2}}.$$

(a) What is $\nabla \times \mathbf{F}$?

(b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ where $0 \le t \le 2\pi$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(c) Is $\mathbf{F}$ conservative on $\mathbb{R}^3 - (z\text{-axis})$?

(d) How would you justify your answer to (c) as quickly as possible?

4. Consider the following vector field with domain $\mathbb{R}^3 - (\{0\} \times [-1, 1] \times \mathbb{R})$:

$$\mathbf{F}(x, y, z) = \frac{(-(y+1), x, 0)}{x^2 + (y+1)^2} - \frac{(-(y-1), x, 0)}{x^2 + (y-1)^2}.$$

- Maybe it helps to see the vector field.
- Go to the applet: `http://www.falstad.com/vector/index.html`.
- Select "user-defined vector field".
- Enter the coordinates lised in the next two steps.
- `- (y+1) / ( x^2 + (y+1)^2 ) + (y-1) / ( x^2 + (y-1)^2 )`
- `x / ( x^2 + (y+1)^2 ) - x / ( x^2 + (y-1)^2 )`
- I have chosen to remove the line segment connecting the two dots from the domain of vector field.

(a) What is $\nabla \times \mathbf{F}$?

(b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (2\cos t, 2\sin t, 0)$ where $-\pi \le t \le \pi$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
Hint: how do $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ and $\mathbf{F}(\mathbf{r}(-t)) \cdot \mathbf{r}'(-t)$ relate?

(c) Is $\mathbf{F}$ conservative on $\mathbb{R}^3 - (\{0\} \times [-1, 1] \times \mathbb{R})$?

(d) Justify your answer to (c).

5. Let $\mathbf{F}_1(x,y,z) = \frac{(-(y+1),x,0)}{x^2+(y+1)^2}$ and $\mathbf{F}_2(x,y,z) = \frac{(-(y-1),x,0)}{x^2+(y-1)^2}$.

   (a) What is $\nabla \times \mathbf{F}_2$? Is $\mathbf{F}_2$ conservative on $\{(x,y,z) \in \mathbb{R}^3 : \ y < \frac{1}{2}\}$?
   Hint: does this domain have holes?
   Let C be parametrized by $\mathbf{r}(t) = (\cos t, -1+\sin t, 0)$, $0 \le t \le 2\pi$.

   (b) What is $\int_C \mathbf{F}_2 \cdot d\mathbf{r}$? Hint: use (a).

   (c) What is $\int_C \mathbf{F}_1 \cdot d\mathbf{r}$?

   (d) Is $\mathbf{F} = \mathbf{F}_1 - \mathbf{F}_2$ conservative on

$$\mathbb{R}^3 - (\{0\} \times \{-1\} \times \mathbb{R}) - (\{0\} \times \{1\} \times \mathbb{R})?$$

   (e) Compare your answer to (d) with question 4(c).

Solutions

1. Consider the following vector field with domain $\mathbb{R}^3-(z\text{-axis})$:

$$\mathbf{F}(x,y,z) = \frac{(-y,x,0)}{\sqrt{x^2+y^2}}.$$

   (a) Direct calculation gives

$$\nabla \times \mathbf{F} = \left(0, 0, \frac{1}{\sqrt{x^2+y^2}}\right).$$

   (b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ where $0 \le t \le 2\pi$. Direct calculation gives $\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$.

   (c) $\mathbf{F}$ is NOT conservative on $\mathbb{R}^3-(z\text{-axis})$.

   (d) I found the line integral to be the quicker calculation.
   A vector field with a non-zero circulation is not conservative.
   The curl calculation also tells you $\mathbf{F}$ is NOT conservative.

2. Consider the following vector field with domain $\mathbb{R}^3-(z\text{-axis})$:

$$\mathbf{F}(x,y,z) = \frac{(-y,x,0)}{x^2+y^2}.$$

   (a) Direct calculation gives $\nabla \times \mathbf{F} = \mathbf{0}$.

   (b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ where $0 \le t \le 2\pi$. Direct calculation gives $\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$.

   (c) $\mathbf{F}$ is NOT conservative on $\mathbb{R}^3-(z\text{-axis})$.

   (d) A vector field with a non-zero circulation is not conservative.

3. Consider the following vector field with domain $\mathbb{R}^3-(z\text{-axis})$:

$$\mathbf{F}(x,y,z) = \frac{(x,y,0)}{\sqrt{x^2+y^2}}.$$

   (a) Direct calculation gives $\nabla \times \mathbf{F} = \mathbf{0}$.

   (b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ where $0 \le t \le 2\pi$. Direct calculation gives $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$.

   (c) $\mathbf{F}$ is conservative on $\mathbb{R}^3-(z\text{-axis})$.

   (d) $f(x,y,z) = \sqrt{x^2+y^2}$ is a potential for $\mathbf{F}$.

4. Consider the following vector field with domain $\mathbb{R}^3 - (\{0\} \times [-1,1] \times \mathbb{R})$:

$$\mathbf{F}(x,y,z) = \frac{(-(y+1), x, 0)}{x^2+(y+1)^2} - \frac{(-(y-1), x, 0)}{x^2+(y-1)^2}.$$

(a) Question 3a together with the chain rule give $\nabla \times \mathbf{F} = \mathbf{0}$.

(b) Let C be the oriented curve parametrized by $\mathbf{r}(t) = (2\cos t, 2\sin t, 0)$ where $-\pi \le t \le \pi$.

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \frac{4+2\sin t}{5+4\sin t} - \frac{4-2\sin t}{5-4\sin t}$$

so $\mathbf{F}(\mathbf{r}(-t)) \cdot \mathbf{r}'(-t) = -\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$. Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

(c) $\mathbf{F}$ is conservative on $\mathbb{R}^3 - (\{0\} \times [-1,1] \times \mathbb{R})$.

(d) $\mathbf{F}$ has zero curl. The domain has a hole, but we checked that the circulation around that hole is zero, so $\mathbf{F}$ is conservative.

Perhaps you'd feel better if we defined a potential for $\mathbf{F}$.
Define a function with domain $\mathbb{R}^3 - (\{0\} \times [-1,1] \times \mathbb{R})$ by

$$f(x,y,z) = \begin{cases} \arctan(\frac{y+1}{x}) - \arctan(\frac{y-1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Direct calculation shows that for $x \neq 0$, $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$.

We worry though. Is f even continuous when $x = 0$?
A point $(0,y,z)$ in the domain has either $y > 1$ or $y < -1$.
Suppose $y > 1$.
Then $\lim_{x\to 0+} f(x,y,z) = \frac{\pi}{2} - \frac{\pi}{2} = 0$
and $\lim_{x\to 0-} f(x,y,z) = -\frac{\pi}{2} + \frac{\pi}{2} = 0$,
so $\lim_{x\to 0} f(x,y,z) = 0 = f(0,y,z)$.
Suppose $y < -1$.
Then $\lim_{x\to 0+} f(x,y,z) = -\frac{\pi}{2} + \frac{\pi}{2} = 0$
and $\lim_{x\to 0-} f(x,y,z) = \frac{\pi}{2} - \frac{\pi}{2} = 0$,
so $\lim_{x\to 0} f(x,y,z) = 0 = f(0,y,z)$.
Yes, f is continuous when $x = 0$.

We worry more. Does $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$ hold when $x=0$?

$$\mathbf{F}(0,y,z) = \left(\frac{2}{y^2-1}, 0, 0\right).$$

We quickly see that $\frac{\partial f}{\partial y}(0,y,z) = \frac{\partial f}{\partial z}(0,y,z) = 0$.
By definition, $\frac{\partial f}{\partial x}(0,y,z) = \lim_{x\to 0} \frac{f(x,y,z)-f(0,y,z)}{x} = \lim_{x\to 0} \frac{f(x,y,z)}{x}$.
By L'Hôpital's rule, $\lim_{x\to 0} \frac{f(x,y,z)}{x} = \lim_{x\to 0} \frac{\partial f}{\partial x}(x,y,z)$.
Finally,

$$\lim_{x\to 0} \frac{\partial f}{\partial x}(x,y,z) = \lim_{x\to 0} \left(\frac{-(y+1)}{x^2+(y+1)^2} - \frac{-(y-1)}{x^2+(y-1)^2}\right) = \frac{2}{y^2-1}.$$

So $\frac{\partial f}{\partial x}(0,y,z) = \frac{2}{y^2-1}$.
Yes, $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$ holds when $x=0$.

Phew!! You can see the end for other approaches.

5. Let $\mathbf{F}_1(x,y,z) = \frac{(-(y+1),x,0)}{x^2+(y+1)^2}$ and $\mathbf{F}_2(x,y,z) = \frac{(-(y-1),x,0)}{x^2+(y-1)^2}$.

   (a) Question 3a together with the chain rule give $\nabla \times \mathbf{F}_2$.
   $\{(x,y,z) \in \mathbb{R}^3 : \ y < \frac{1}{2}\}$ has no holes,
   so $\mathbf{F}_2$ is conservative on $\{(x,y,z) \in \mathbb{R}^3 : \ y < \frac{1}{2}\}$.
   Let C be parametrized by $\mathbf{r}(t) = (\cos t, -1+\sin t, 0)$, $0 \le t \le 2\pi$.

   (b) $\int_C \mathbf{F}_2 \cdot d\mathbf{r} = 0$ since $\mathbf{F}_2$ is conservative on $\{(x,y,z) \in \mathbb{R}^3 : \ y < \frac{1}{2}\}$.

   (c) Direct calculation gives $\int_C \mathbf{F}_1 \cdot d\mathbf{r} = 2\pi$.

   (d) Let $\mathbf{F} = \mathbf{F}_1 - \mathbf{F}_2$.
   $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ so $\mathbf{F}$ has a non-zero circulation on

   $$\mathbb{R}^3 - (\{0\} \times \{-1\} \times \mathbb{R}) - (\{0\} \times \{1\} \times \mathbb{R})$$

   which means it is NOT conservative on this domain.
   Notice that trying to define a potential as in 4(d) would result in a discontinuous function since

   $$\lim_{x\to 0+} f(x,0,0) = \pi \neq -\pi = \lim_{x\to 0-} f(x,0,0).$$

   (e) Putting a wall between $\{0\} \times \{-1\} \times \mathbb{R}$ and $\{0\} \times \{1\} \times \mathbb{R}$), makes $\mathbf{F}$ change from not conservative to conservative.
   $\mathbf{F}$ "detects" loops which are nontrivial and pass through the region being blocked by the wall.

6. Another formula for the potential I gave in 4.(d) is

$$\operatorname{sgn}(x)\left[\arcsin\left(\frac{y+1}{\sqrt{x^2+(y+1)^2}}\right)-\arcsin\left(\frac{y-1}{\sqrt{x^2+(y-1)^2}}\right)\right].$$

Another way to prove that f is differentiable on the plane $x = 0$ is to note that for $|y| > 1$, we have

$$f(x,y,z) = \arctan\left(\frac{x}{y-1}\right) - \arctan\left(\frac{x}{y+1}\right).$$

This formula is not valid when $-1 \leq y \leq 1$. We have

$$f(x,y,z) = \begin{cases} \arctan(\frac{x}{y-1}) - \arctan(\frac{x}{y+1}) + \pi & \text{if } x > 0,\ -1 < y < 1 \\ \arctan(\frac{x}{y-1}) - \arctan(\frac{x}{y+1}) - \pi & \text{if } x < 0,\ -1 < y < 1. \end{cases}$$

7. How do question 4 and 5 change for the domain

$$\mathbb{R}^3 - (\{0\} \times (-\infty, -1] \times \mathbb{R}) - (\{0\} \times [1, \infty) \times \mathbb{R})?$$