Math 31B
Integration and Infinite Series

Midterm 2

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 48 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ______________________________

Discussion: ______________________________

Question	Points	Score
1	12	
2	10	
3	12	
4	14	
Total:	48	

Problem 1.

(a) [2pts.] Give the value or say, "undefined."

$$\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right) =$$

Solution: $\frac{\pi}{3}$.

(b) [2pts.] Give the value or say, "undefined."

$$\sin\left(\arcsin\left(\frac{\pi}{2}\right)\right) =$$

Solution: Undefined.

(c) [2pts.] Calcluate $\frac{d}{dx}(\sin(\arcsin x))$.

Solution: $= \frac{d}{dx}(x) = 1$.

(d) [6pts.] Show that

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{3-2x^2}}\, dx = \frac{\pi}{4\sqrt{2}}.$$

Solution: Let $u = \sqrt{\frac{2}{3}} \cdot x$. Then $2x^2 = 3u^2$ and $dx = \sqrt{\frac{3}{2}} \cdot du$. So

$$\begin{aligned}
\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{3-2x^2}}\, dx = \int_0^{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{\sqrt{3}}} \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{1}{\sqrt{3-3u^2}}\, du &= \frac{1}{\sqrt{2}} \int_0^{\frac{1}{\sqrt{2}}} \cdot \frac{1}{\sqrt{1-u^2}}\, du \\
&= \frac{1}{\sqrt{2}} \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4\sqrt{2}}.
\end{aligned}$$

Problem 2. *10pts.*

Calculate the following indefinite integral.

$$\int \frac{2x^3 + 2x^2 - 2x + 1}{x^2(x-1)^2}\, dx$$

So you know that you're going in the right direction: one of the coefficients in the partial fraction decomposition is 0. I found that the x and x^3 coefficients were the quickest to calculate when solving for the constants.

Solution: First, we find A, B, C, and D with

$$\frac{2x^3 + 2x^2 - 2x + 1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2},$$

i.e. $2x^3 + 2x^2 - 2x + 1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$.

Letting $x = 0$ we see $B = 1$. Letting $x = 1$ we see $D = 3$.

Comparing the coefficients of x we see $A - 2B = -2$, so $A = 0$.

Comparing the coefficients of x^3 we see $A + C = 2$, so $C = 2$.

Thus,

$$\begin{aligned}\int \frac{2x^3 + 2x^2 - 2x + 1}{x^2(x-1)^2}\, dx &= \int \frac{1}{x^2} + \frac{2}{x-1} + \frac{3}{(x-1)^2}\, dx \\ &= -\frac{1}{x} + 2\ln|x-1| - \frac{3}{x-1} + c.\end{aligned}$$

Problem 3.

In each case, say whether the series converges absolutely, converges conditionally, or diverges. You should justify your answer carefully using relevant tests.

(a) [4pts.] $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

Solution: Let $a_n = \frac{1}{\sqrt{n}}$. Then (a_n) is decreasing and $\lim_{n\to\infty} a_n = 0$.
The alternating series test says $\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.
Since $\frac{1}{2} \leq 1$, the p-test says that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.
Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

(b) [4pts.] $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$.

Solution: Let $a_n = \frac{(-1)^n}{(2n)!}$. Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0 < 1.$$

The ratio test says that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ converges absolutely.

(c) [4pts.] $\sum_{n=1}^{\infty} \left(\frac{8n+18}{19n-88}\right)^n$.

Solution: Let $a_n = \left(\frac{8n+18}{19n-88}\right)^n$. Then

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{8n+18}{|19n-88|} = \frac{8}{19} < 1.$$

The root test says that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{8n+18}{19n-88}\right)^n$ converges absolutely.

Problem 4.

(a) [2pts.] Write down $T_3(x)$ centered at 0 for $f(x) = 9 + 8x + 7x^2 + 6x^4 + x^5$.

> **Solution:** $9 + 8x + 7x^2$.

(b) [2pts.] Write down $T_{10}(x)$ centered at 0 for $f(x) = 9 + 8x + 7x^2 + 6x^4 + x^5$.

> **Solution:** $9 + 8x + 7x^2 + 6x^4 + x^5$.

(c) [2pts.] Write down $T_5(x)$ centered at 3 for $f(x) = \sin(x - 3)$.

> **Solution:** $(x-3) - \frac{(x-3)^3}{3!} + \frac{(x-3)^5}{5!}$.

(d) [2pts.] (More difficult. Don't waste time.)
Write down $T_{101}(x)$ centered at 0 for $f(x) = x^{98}e^x$.

> **Solution:** $x^{98} + x^{99} + \frac{x^{100}}{2!} + \frac{x^{101}}{3!}$.

(e) [6pts.] Suppose $f(x)$ is a function with

$$|f^{(n+1)}(u)| \leq \frac{n^2}{u^{n+1}} \quad \text{for all } u > 0.$$

Let $T_n(x)$ be the n-th Taylor polynomial of $f(x)$ centered at 1.
Find n so that

$$\left| f\left(\frac{1}{2}\right) - T_n\left(\frac{1}{2}\right) \right| \leq \frac{1}{10^8}.$$

Solution: First, a silly solution I didn't plan on...
When $n = 0$, the given equation says $|f'(u)| \leq 0$ for all $u > 0$. So $f'(u) = 0$ for all $u > 0$. So f is constant for $u > 0$. So $f(\frac{1}{2}) = T_0(\frac{1}{2})$, and $n = 0$ works.

Now, the intended solution...
Since $\frac{n^2}{u^{n+1}}$ is decreasing, we can take $K = \frac{n^2}{(\frac{1}{2})^{n+1}}$, and Taylor's error bound says

$$\left| f\left(\frac{1}{2}\right) - T_n\left(\frac{1}{2}\right) \right| \leq \frac{\frac{n^2}{(\frac{1}{2})^{n+1}} \cdot |\frac{1}{2} - 1|^{n+1}}{(n+1)!} = \frac{n^2}{(n+1)!}.$$

When $n = 10^8 + 1$ we have

$$\frac{n^2}{(n+1)!} \leq \frac{n(n+1)}{(n+1)!} = \frac{1}{(n-1)!} \leq \frac{1}{n-1} = \frac{1}{10^8},$$

so

$$\left| f\left(\frac{1}{2}\right) - T_{10^8+1}\left(\frac{1}{2}\right) \right| \leq \frac{1}{10^8}.$$