TAYLOR POLYNOMIALS

SUPPOSE YOU KNOW A FUNCTION IS DEFINED BY A POLYNOMIAL OF DEGREE ≤ 5

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

AND THAT
$$\begin{aligned} f(0) &= 4 \\ f'(0) &= 3 \\ f''(0) &= 9 \\ f'''(0) &= 5 \\ f''''(0) &= 11 \\ f'''''(0) &= 7. \end{aligned}$$

CAN YOU TELL ME THE POLYNOMIAL?

USING $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

WE GET
$$\begin{aligned} f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \\ f''(x) &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 \\ f'''(x) &= 6a_3 + 24a_4 x + 60a_5 x^2 \\ f''''(x) &= 24a_4 + 120a_5 x \\ f'''''(x) &= 120a_5 \end{aligned}$$

So
$$\begin{aligned} f(0) &= a_0 = 4 \\ f'(0) &= a_1 = 3 \\ f''(0) &= 2a_2 = 9 \\ f'''(0) &= 6a_3 = 5 \\ f''''(0) &= 24a_4 = 11 \\ f'''''(0) &= 120a_5 = 7 \end{aligned}$$

So
$$f(x) = 4 + \frac{3}{1}x + \frac{9}{2}x^2 + \frac{5}{6}x^3 + \frac{11}{24}x^4 + \frac{7}{120}x^5.$$

SUPPOSE YOU KNOW A FUNCTION IS DEFINED BY A POLYNOMIAL OF DEGREE ≤ 3

AND THAT $f($[redacted]$) = 20$

$$f'(1) = 2$$
$$f''(1) = 7$$
$$f'''(1) = 5$$

CAN YOU TELL ME THE POLYNOMIAL?

THE TRICK:

$$f(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$$

So
$$f'(x) = a_1 + 2a_2(x-1) + 3a_3(x-1)^2$$
$$f''(x) = 2a_2 + 6a_3(x-1)$$
$$f'''(x) = 6a_3$$

So
$$f(1) = a_0 = 20$$
$$f'(1) = a_1 = 2$$
$$f''(1) = 2a_2 = 7$$
$$f'''(1) = 6a_3 = 5$$

So $f(x) = 20 + \frac{2}{1}(x-1) + \frac{7}{2}(x-1)^2 + \frac{5}{6}(x-1)^3.$

SUPPOSE YOU KNOW A FUNCTION IS DEFINED BY A POLYNOMIAL OF DEGREE $\leq n$

AND YOU KNOW $f(a), f'(a), f''(a), \ldots, f^{(n)}(a)$.

CAN YOU TELL ME THE POLYNOMIAL?

$$f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$
$$+ \frac{f^{(4)}(a)}{24}(x-a)^4 + \frac{f^{(5)}(a)}{120}(x-a)^5 + \ldots$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$n! = n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1.$

IF $f(x)$ IS A FUNCTION

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

IS CALLED THE

n-TH TAYLOR POLYNOMIAL OF f CENTERED AT a.

THE POINT OF THIS:

- IF $f(x)$ IS A POLYNOMIAL OF DEGREE $\leq n$ WE GET $f(x)$ BACK.
- IF $f(x)$ IS <u>NOT</u> A POLYNOMIAL OF DEGREE $\leq n$ WE OBTAIN A

  USEFUL APPOXIMATION OF $f(x)$ NEAR a

EXAMPLE (THE TAYLOR POLYNOMIALS OF e^x CENTERED AT 0)

LET $f(x) = e^x$.

WE'LL CALCULATE THE TAYLOR POLYNOMIALS CENTERED AT 0.

$f'(x) = e^x$
$f'(0) = 1$

$f''(x) = e^x$
$f''(0) = 1$

$f'''(x) = e^x$
$f'''(0) = 1$

$$T_0(x) = f(0) = 1$$

$$T_1(x) = f(0) + \frac{f'(0)}{1}x = 1 + x$$

$$T_2(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 = 1 + x + \frac{x^2}{2}$$

$$T_3(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$$

$$T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}$$

LAST TIME:

IF $f(x)$ IS A FUNCTION
(WHICH IS n TIMES DIFFERENTIABLE AT a)

$$\boxed{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

IS CALLED THE

n-TH TAYLOR POLYNOMIAL OF f
CENTERED AT a.

$\left(n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\right)$

THE POINT OF THIS:

- IF $f(x)$ IS A POLYNOMIAL OF DEGREE $\leq n$ WE GET $f(x)$ BACK
- IF $f(x)$ IS NOT A POLYNOMIAL OF DEGREE $\leq n$ WE OBTAIN A

$$\boxed{\text{USEFUL APPROXIMATION OF } f(x) \text{ NEAR } a.}$$

EXAMPLE (e^x AT 0)

THE n-TH TAYLOR POLYNOMIAL OF e^x CENTERED AT 0 IS

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

EXAMPLE (THE TAYLOR POLYNOMIALS OF $\sin x$ CENTERED AT 0)

LET $f(x) = \sin x$ $\qquad f(0) = 0$

THEN
$$f'(x) = \cos x \qquad f'(0) = 1$$
$$f''(x) = -\sin x \qquad f''(0) = 0$$
$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4n)}(x) = \sin x \qquad f^{(4n)}(0) = 0$$
$$f^{(4n+1)}(x) = \cos x \qquad f^{(4n+1)}(0) = 1$$
$$f^{(4n+2)}(x) = -\sin x \qquad f^{(4n+2)}(0) = 0$$
$$f^{(4n+3)}(x) = -\cos x \qquad f^{(4n+3)}(0) = -1$$

SO
$$T_0(x) = f(0) = 0$$
$$T_1(x) = f(0) + \frac{f'(0)}{1}x = 0 + x$$

$$T_2(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 = 0 + x + 0x^2$$

$$T_3(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$
$$= 0 + x + 0x^2 - \frac{x^3}{6}$$

$$T_4(x) = 0 + x + 0x^2 - \frac{x^3}{6} + 0x^4$$
$$T_5(x) = 0 + x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120}$$

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WE FIND THAT THE TAYLOR POLYNOMIALS FOR $\sin x$ CENTERED AT 0 ARE GIVEN BY

$$T_{2n+1}(x) = T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

PLOT TAYLOR POLYNOMIALS FOR

e^x AND $\sin x$

NEXT TO ACTUAL FUNCTIONS.

$\sin x$ IS DRAWN HERE:

www.desmos.com/calculator/noanuckuli

Q: SUPPOSE f IS A FUNCTION AND THAT T_n IS THE n-TH TAYLOR POLYNOMIAL OF f CENTERED AT a.

FOR A GIVEN x, HOW BIG DOES n NEED TO BE FOR $T_n(x)$ TO APPROXIMATE $f(x)$ WELL?

LET $E_n(x) = f(x) - T_n(x)$, THE ERROR BETWEEN f AND T_n AT x.

WE ARE ASKING, "FOR A GIVEN x, HOW BIG DOES n NEED TO BE FOR $E_n(x)$ TO BE SMALL?"

A: TAYLOR'S ERROR BOUND

f SUPPOSE f IS A FUNCTION (WHICH IS $(n+1)$-TIMES DIFFERENTIABLE)

n AND THAT T_n IS THE n-TH TAYLOR
a POLYNOMIAL OF f CENTERED AT a.

x LET $E_n(x) = f(x) - T_n(x)$ BE THE ERROR AT x.

K **IF** $\left|f^{(n+1)}(u)\right| \leq K$ FOR ALL u BETWEEN a AND x

THEN $\left|E_n(x)\right| \leq \dfrac{K|x-a|^{n+1}}{(n+1)!}$.

THIS LECTURE

\+ THINKING !

WE'LL DO THIS BIT LOTS NEXT TIME.

THE FIRST PART OF THE THEOREM TELLS YOU WHO EVERYONE IS:

f, n, a, AND x.

THE SECOND PART IS AN "IF, THEN" SENTENCE. TO CONCLUDE THE BIT AFTER "THEN" YOU MUST CHOOSE K SO THAT THE BIT AFTER THE "IF" IS TRUE.

A SUCCESSFUL APPLICATION OF TAYLOR'S ERROR BOUND WILL RESULT IN AN INEQUALITY

$$\left|E_n(x)\right| \leq \frac{K|x-a|^{n+1}}{(n+1)!}$$

THE RHS MIGHT BE BIG, IN WHICH CASE APPLYING THE THEOREM WAS NOT SO HELPFUL. THIS IS WHERE CHOOSING n COMES IN. YOU CAN APPLY THE THEOREM FOR MANY DIFFERENT n, UNTIL YOU GET A USEFUL INEQUALITY.

EXAMPLE (READING TAYLOR'S ERROR BOUND IN A SPECIFIC CASE)

LET $T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$,

THE FOURTH TAYLOR POLYNOMIAL OF e^x ABOUT 0.

ESTIMATE $\left| e^{-1} - T_4(-1) \right|$ USING TAYLOR'S ERROR BOUND.

READING THE THEOREM WITH

$$f(u) = e^u, \quad n = 4, \quad a = 0, \quad x = -1$$

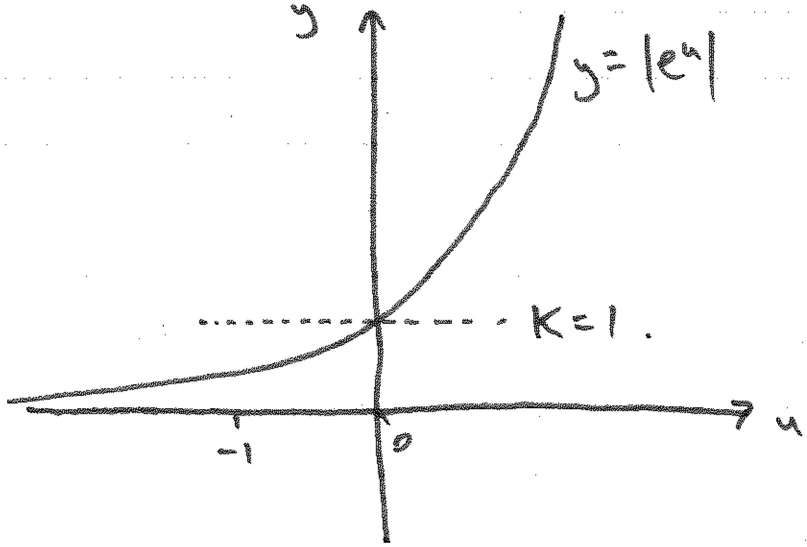
IT SAYS $\qquad f^{(5)}(u)$

" IF $\left| e^u \right| \leq K$ FOR ALL u BETWEEN 0 AND -1

THEN $$\left| e^{-1} - T_4(-1) \right| \leq \frac{K \left| (-1) - 0 \right|^5}{5!}$$

i.e. $$\left| e^{-1} - T_4(-1) \right| \leq \frac{K}{5!}. \text{ "}$$

DRAW THE GRAPH OF $y = |e^u|$



SINCE $|e^u| \leq 1$ FOR ALL u BETWEEN 0 AND -1

WE OBTAIN $\left| e^{-1} - T_4(-1) \right| \leq \frac{1}{5!} = \frac{1}{120}$.

NOTICE : IF WE'D HAVE STARTED WITH T_n INSTEAD OF T_4 WE WOULD HAVE GOT

$$\left| e^{-1} - T_n(-1) \right| \leq \frac{1}{(n+1)!}.$$

LAST TIME: TAYLOR'S ERROR BOUND

f SUPPOSE f IS A FUNCTION (WHICH IS $(n+1)$-TIMES DIFFERENTIABLE)

n AND THAT T_n IS THE n-TH TAYLOR POLYNOMIAL
a OF f CENTERED AT a.

x LET $E_n(x) = f(x) - T_n(x)$ BE THE ERROR AT x.

K IF $\left|f^{(n+1)}(u)\right| \leq K$ FOR ALL u BETWEEN a AND x,

THEN $\left|E_n(x)\right| \leq \dfrac{K|x-a|^{n+1}}{(n+1)!}$

EXAMPLE (READING TAYLOR'S ERROR BOUND IN A SPECIFIC CASE)

LET $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$,

THE THIRD TAYLOR POLYNOMIAL OF $\ln x$ ABOUT 1.

ESTIMATE $\left| \ln\left(\frac{13}{10}\right) - T_3\left(\frac{13}{10}\right) \right|$ USING TAYLOR'S ERROR BOUND.

LET $f(u) = \ln u$.

THEN $f'(u) = \frac{1}{u}$, $\quad f''''(u) = \frac{-6}{u^4}$.

$f''(u) = \frac{-1}{u^2}$,

$f'''(u) = \frac{2}{u^3}$,

So, READING THE THEOREM WITH

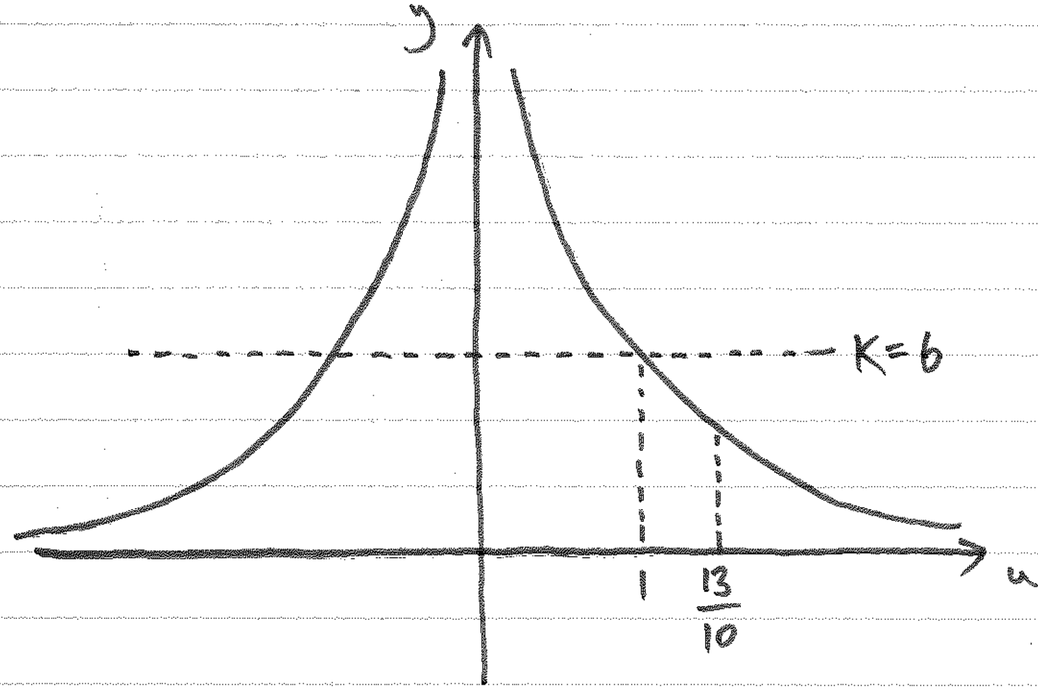
$$f(u) = \ln u, \quad n = 3, \quad a = 1, \quad x = \frac{13}{10}$$

IT SAYS " IF $\left| \frac{6}{u^4} \right| \leq K$ FOR ALL u BETWEEN 1 AND $\frac{13}{10}$,

THEN $\left| \ln\left(\frac{13}{10}\right) - T_3\left(\frac{13}{10}\right) \right| \leq \frac{K\left|\frac{13}{10} - 1\right|^4}{4!}$

i.e. $\left| \ln\left(\frac{13}{10}\right) - T_3\left(\frac{13}{10}\right) \right| \leq \frac{27K}{80{,}000}$ ".

DRAW GRAPH OF $y = \left|\frac{6}{u^4}\right|$



SINCE $\left|\frac{6}{u^4}\right| \leq 6$ FOR ALL u BETWEEN 1 AND $\frac{13}{10}$,

WE OBTAIN

$$\left|\ln\left(\frac{13}{10}\right) - T_3\left(\frac{13}{10}\right)\right| \leq \frac{81}{40,000} = 0.002025$$

(BY TAKING $K = 6$)

EXAMPLE (READING TAYLOR'S ERROR BOUND IN A SPECIFIC CASE)

LET $T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$,

THE FOURTH TAYLOR POLYNOMIAL OF $\ln x$ ABOUT 1.

ESTIMATE $\left| \ln\left(\frac{13}{10}\right) - T_4\left(\frac{13}{10}\right) \right|$ USING TAYLOR'S ERROR BOUND.

LET $f(u) = \ln u$, THEN $f^{(5)}(u) = \frac{24}{u^5}$.

SO READING THE THEOREM WITH

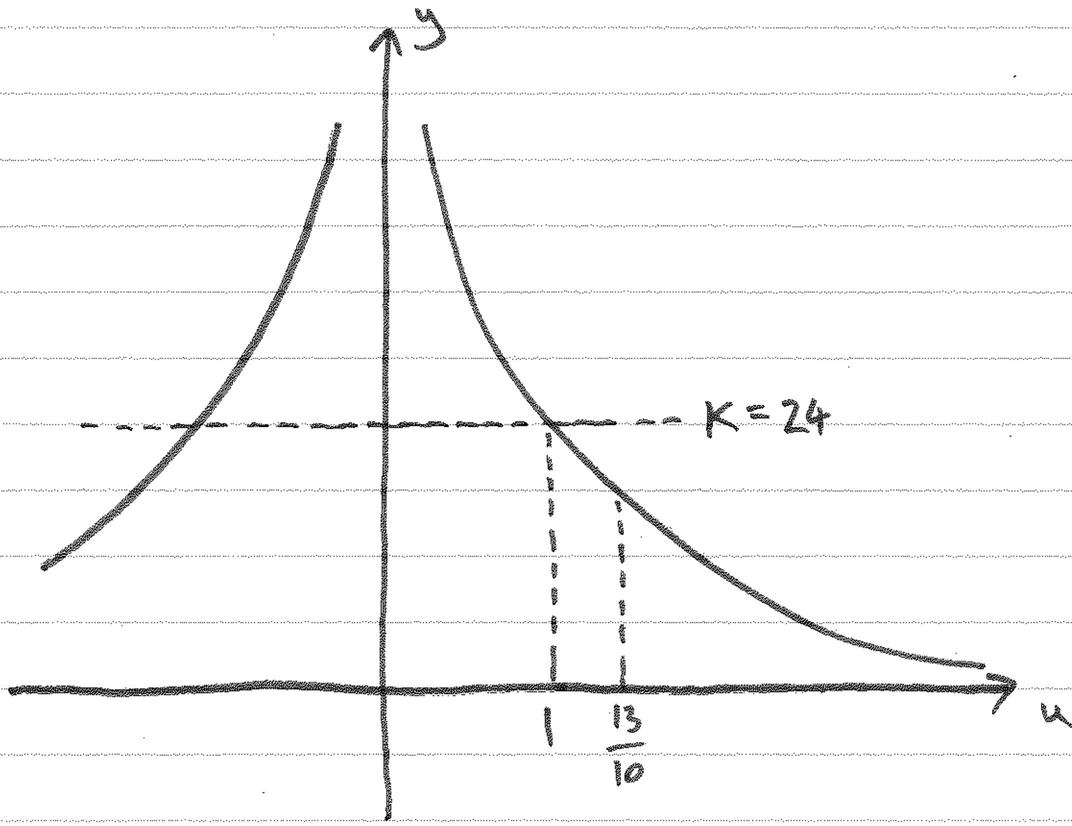
$$f(u) = \ln u, \quad n=4, \quad a=1, \quad x = \frac{13}{10}$$

IT SAYS "IF $\left|\frac{24}{u^5}\right| \leq K$ FOR ALL u BETWEEN 1 AND $\frac{13}{10}$,

$$\text{THEN} \quad \left| \ln\left(\frac{13}{10}\right) - T_4\left(\frac{13}{10}\right) \right| \leq \frac{K\left|\frac{13}{10}-1\right|^5}{5!}$$

$$\text{i.e.} \quad \left| \ln\left(\frac{13}{10}\right) - T_4\left(\frac{13}{10}\right) \right| \leq \frac{81K}{4\cdot 10^6} \text{"}$$

DRAW GRAPH OF $y = \left|\frac{24}{u^5}\right|$



SINCE $\left|\frac{24}{u^5}\right| \leq 24$ FOR ALL u BETWEEN 1 AND $\frac{13}{10}$,

WE OBTAIN

$$\left| \ln\left(\frac{13}{10}\right) - T_4\left(\frac{13}{10}\right) \right| \leq \frac{486}{10^6} = 0.000486.$$

(BY TAKING $K = 24$)

EXAMPLE

LET $T_n(x)$ BE THE n-TH TAYLOR POLYNOMIAL OF $\ln x$ CENTERED AT $x=1$.

FIND AN n SO THAT

$$\left| \ln\left(\frac{13}{10}\right) - T_n\left(\frac{13}{10}\right) \right| \leq 0.002.$$

WE JUST SAW THAT

$$\left| \ln\left(\frac{13}{10}\right) - T_4\left(\frac{13}{10}\right) \right| \leq 0.000486 \leq 0.002$$

SO $n=4$ WORKS.

IN FACT,

$$\left| \ln\left(\frac{13}{10}\right) - T_3\left(\frac{13}{10}\right) \right| \leq 0.002$$

BUT WE DO <u>NOT</u> LEARN THIS FROM THE ERROR <u>BOUND</u>; WE ONLY GOT ≤ 0.002025.

EXAMPLE:

CALCULATE $\cos\left(\frac{3}{4}\right)$ TO WITHIN 0.01 WITHOUT A CALCULATOR.

REPHRASING:

LET T_n BE THE n-TH TAYLOR POLYNOMIAL OF $\cos$ CENTERED AT 0.

FIND AN n SO THAT

$$\left| \cos\left(\frac{3}{4}\right) - T_n\left(\frac{3}{4}\right) \right| \leq 0.01.$$

LET $f(u) = \cos u$, n UNSPECIFIED, $a = 0$, $x = \frac{3}{4}$.

TAYLOR'S ERROR BOUND SAYS

"IF $\left|\cos^{(n+1)}(u)\right| \leq K$ FOR ALL u BETWEEN 0 AND $\frac{3}{4}$,

THEN $$\left|\cos\left(\frac{3}{4}\right) - T_n\left(\frac{3}{4}\right)\right| \leq \frac{K\left|\frac{3}{4} - 0\right|^{n+1}}{(n+1)!}$$

$$\leq \frac{K}{(n+1)!}."$$

SINCE $\left|\cos^{(n+1)}(u)\right| \leq 1$ FOR ALL u

WE GET $$\left|\cos\left(\frac{3}{4}\right) - T_n\left(\frac{3}{4}\right)\right| \leq \frac{1}{(n+1)!}.$$

WHEN $n = 4$, THIS GIVES

$$\left|\cos\left(\frac{3}{4}\right) - T_4\left(\frac{3}{4}\right)\right| \leq \frac{1}{5!} \leq 0.01.$$

SO $$\cos\left(\frac{3}{4}\right) = T_4\left(\frac{3}{4}\right) = 1 - \frac{\left(\frac{3}{4}\right)^2}{2!} + \frac{\left(\frac{3}{4}\right)^4}{4!}$$

TO WITHIN 0.01.