PARTIAL FRACTIONS

SOME INTEGRALS WE'VE SEEN

WHEN $n \neq -1$

$$\int (x+a)^n \, dx = \frac{(x+a)^{n+1}}{n+1} + c$$

$n=-2$ SAYS / ALSO,

$$\boxed{\begin{aligned} &\int \frac{1}{(x+a)^2}\,dx = \frac{-1}{x+a} + c \\ &\int \frac{1}{x+a}\,dx = \ln|x+a| + c \end{aligned}}$$

WHEN $n \neq -1$

$$\int x(x^2+a^2)^n \, dx = \frac{(x^2+a^2)^{n+1}}{2(n+1)} + c$$

$n=-2$ SAYS / ALSO,

$$\boxed{\begin{aligned} &\int \frac{x}{(x^2+a^2)^2}\,dx = \frac{-1}{2(x^2+a^2)} + c \\ &\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + c. \end{aligned}}$$

WHEN $a \neq 0$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,\mathrm{ARCTAN}\left(\frac{x}{a}\right) + c.}$$

CHALLENGE : $\int (x^2+a^2)^n \, dx$ FOR $n < -1$.

WE WON'T DO THIS.

GOAL : $\int \frac{p(x)}{q(x)} \, dx$

FOR ANY POLYNOMIALS $p(x), q(x)$.

IDEA : TRY TO REDUCE TO THE INTEGRALS WE HAVE JUST HIGHLIGHTED.

EXAMPLE (NON-REPEATED LINEAR FACTORS IN DENOMINATOR)

CALCULATE $\displaystyle\int \frac{6x^2 - x - 1}{x(x-1)(x+1)}\,dx$.

STEP 1 (CORRECT PARTIAL FRACTION DECOMPOSITION):

$$\frac{6x^2 - x - 1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

STEP 2 (CLEAR DENOMINATORS)

$$(*) \quad 6x^2 - x - 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1).$$

STEP 3 (FIGURE OUT VALUES OF CONSTANTS)

IN THIS EXAMPLE,
SETTING $x = 0, 1, -1$, IN TURN IS USEFUL.

SETTING $x=0$ IN $(*)$ GIVES $-1 = -A$
" $x = 1$ " $(*)$ " $4 = 2B$
" $x = -1$ " $(*)$ " $6 = 2C$

So $A = 1$, $B = 2$, $C = 3$.

STEP 4 (DO INTEGRAL):
$$\int \frac{6x^2 - x - 1}{x(x-1)(x+1)}\,dx = \int \frac{1}{x} + \frac{2}{x-1} + \frac{3}{x+1}\,dx = \ln|x| + 2\ln|x-1| + 3\ln|x+1| + C.$$

EXAMPLE (ONE LINEAR, ONE IRREDUCIBLE QUADRATIC, NOT REPEATED)

CALCULATE $\displaystyle\int \frac{2x^2-2x+4}{x(x^2+4)}\,dx$

STEP 1 (CORRECT PARTIAL FRACTION DECOMPOSITION):

$$\frac{2x^2-2x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

STEP 2 (CLEAR DENOMINATORS)

$$(*) \qquad 2x^2-2x+4 = A(x^2+4) + (Bx+C)x$$

STEP 3 (FIGURE OUT VALUES OF CONSTANTS)

IN THIS EXAMPLE,
COMPARING COEFFICIENTS IS USEFUL.

MULTIPLY (*) OUT GIVES

$$2x^2-2x+4 = (A+B)x^2 + Cx + 4A$$

$$2 = A+B \qquad -2 = C \qquad 4 = 4A$$

SO $A=1$, $B=1$, $C=-2$

STEP 4 (DO INTEGRAL): $\displaystyle\int \frac{2x^2-2x+4}{x(x^2+4)}\,dx = \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{2}{x^2+4}\,dx = \ln|x| + \frac{1}{2}\ln(x^2+4) - \text{ARCTAN}\left(\frac{x}{2}\right) + C$

EXAMPLE (TWO LINEAR, ONE IRREDUCIBLE QUADRATIC, NOT REPEATED)

CALCULATE $\displaystyle\int \frac{6x^3-x-1}{x(x-1)(x^2+1)}\,dx$

STEP 1: $$\frac{6x^3-x-1}{x(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}.$$

STEP 2: $$6x^3-x-1 = A(x-1)(x^2+1) + Bx(x^2+1) + (Cx+D)x(x-1). \qquad (*)$$

STEP 3: IN THIS EXAMPLE, BOTH OF THE PREVIOUSLY ENCOUNTERED TECHNIQUES ARE USEFUL.

SETTING $x=0$ IN $(*)$ GIVES $-1=-A$
" $x=1$ " $(*)$ " $4=2B$

SETTING $x=-1$ IN $(*)$ GIVES
$$-6 = -4A-2B-2C+2D$$

SO $A=1$, $B=2$, AND $D=1+C$.

LOOKING AT x^3 COEFF. GIVES $6=A+B+C$,
SO $C=3$, $D=4$.

STEP 4 :

$$\int \frac{6x^3 - x - 1}{x(x-1)(x^2+1)}\, dx$$

$$= \int \frac{1}{x} + \frac{2}{x-1} + \frac{3x}{x^2+1} + \frac{4}{x^2+1}\, dx$$

$$= \ln|x| + 2\ln|x-1| + \frac{3}{2}\ln(x^2+1) + 4\text{ARCTAN}(x) + c$$

LAST TIME:

1) $$\int \frac{6x^2-x-1}{x(x-1)(x+1)}\,dx = \int \frac{1}{x} + \frac{2}{x-1} + \frac{3}{x+1}\,dx = \ldots$$

2) $$\int \frac{2x^2-2x+4}{x(x^2+4)}\,dx = \int \frac{1}{x} + \frac{x-2}{x^2+4}\,dx = \ldots$$

3) $$\int \frac{6x^3-x-1}{x(x-1)(x^2+1)}\,dx = \int \frac{1}{x} + \frac{2}{x-1} + \frac{3x+4}{x^2+1}\,dx = \ldots$$

1) DEMONSTRATED SETTING x TO BE CERTAIN VALUES IN ORDER TO FIND CONSTANTS IN PARTIAL FRACTION DECOMPOSITION

2) DEMONSTRATED EQUATING COEFFICIENTS
3) DEMONSTRATED COMBINING TECHNIQUES.

Q: HOW DO WE KNOW WHAT PARTIAL FRACTION DECOMPOSITION TO USE?

FIRST, A <u>FACT</u>.

GIVEN A POLYNOMIAL $p(x)$, YOU CAN ALWAYS FACTOR IT INTO

1) LINEAR TERMS

$$x+a$$

2) QUADRATIC TERMS WHICH DON'T FACTOR FURTHER

$$x^2+bx+c$$

WHERE $b^2-4c<0$

IN EXAMPLES SO FAR, THE DENOMINATOR WAS ALREADY IN THIS FORM.

YOU <u>MUST</u> PUT DENOMINATOR IN THIS FORM IF IT IS NOT ALREADY.

PARTIAL FRACTIONS W/ NON-REPEATED FACTORS

IF $\deg p(x) < \deg q(x)$ AND

$$q(x) = (x+a_1)\cdots(x+a_m)(x^2+b_1x+c_1)\cdots(x^2+b_nx+c_n)$$

DON'T FACTOR FURTHER

ALL DIFFERENT

THEN WE CAN WRITE

$$\frac{p(x)}{q(x)} = \frac{A_1}{x+a_1} + \cdots + \frac{A_m}{x+a_m}$$

$$+ \frac{B_1x+C_1}{x^2+b_1x+c_1} + \cdots + \frac{B_nx+C_n}{x^2+b_nx+c_n}$$

PARTIAL FRACTIONS W/ REPEATED FACTORS

AGAIN, SUPPOSE $\deg p(x) < \deg q(x)$. $\frac{p(x)}{q(x)}$?

Q: WHAT IF HAVE $(x+a)^r$ INSTEAD OF $(x+a)$?

ie. WHAT IF $(x+a)$ IS A REPEATED LINEAR FACTOR?

A: INSTEAD OF $\frac{A}{x+a}$ HAVE

$$\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \dots + \frac{A_r}{(x+a)^r}.$$

Q: WHAT IF HAVE $(x^2+bx+c)^s$ INSTEAD OF x^2+bx+c?

ie (x^2+bx+c) IS A REPEATED QUADRATIC FACTOR?

A: INSTEAD OF $\frac{Bx+C}{x^2+bx+c}$ HAVE

$$\frac{B_1x+C_1}{x^2+bx+c} + \dots + \frac{B_sx+C_s}{(x^2+bx+c)^s}.$$

EXAMPLE (REPEATED LINEAR FACTOR IN DENOMINATOR)

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx$$

STEP 1 : $\dfrac{4x}{(x-1)^2(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x+1)}$

STEP 2 : $4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

STEP 3 : LETTING $x=1$ GIVES $4=2B$.
LETTING $x=-1$ GIVES $-4=4C$.

SO $B=2$, $C=-1$.

TRY LETTING $x=0$,

$$0 = -A + B + C$$
$$A = B + C = 1$$

STEP 4 : $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx =$

$$\int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\,dx =$$

$$\ln|x-1| - \frac{2}{x-1} - \ln|x+1| + c.$$

EXAMPLE (REPEATED QUADRATIC FACTOR IN DENOMINATOR)

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$$

STEP 1: $\dfrac{1-x+2x^2-x^3}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

STEP 2: $1-x+2x^2-x^3 =$
$A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$

STEP 3: (MORE DIFFICULT)
EXPANDING GIVES
$1-x+2x^2-x^3 =$
$A+(C+E)x+?x^2+Cx^3+(A+B)x^4$

x^0: $A=1$
x^4: $A+B=0$ so $B=-1$
x^3: $C=-1$
x: $C+E=-1$ so $E=0$

USING x^2 CAN GET $D=1$
OR LETTING $x=1$ GIVES
$1 = 4A + (B+C)\cdot 2 + (D+E)$
$1 = 4-4+D+0 = D$, $D=1$.

STEP 4:

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \mathrm{ARCTAN}(x) - \frac{1}{2(x^2+1)} + c.$$

LAST TIME:

LEARNED HOW TO DO

$$\int \frac{p(x)}{q(x)}\,dx$$

WHEN $\deg p(x) < \deg q(x)$

AND
$$q(x) = (x+a_1)^{r_1} \cdots (x+a_m)^{r_m} \cdot (x^2+b_1^2)^{s_1} \cdots (x^2+b_n^2)^{s_n}.$$

SAW LOTS OF EXAMPLES.

A COUPLE OF PROBLEMS ARE LEFT OVER...

EXAMPLE (COMPLETING THE SQUARE)

$$\int \frac{2x+3}{x^2+2x+2}\,dx$$

- "b^2-4c" $= 4-8 = -4 < 0$,
  SO THE QUADRATIC DOES NOT FACTOR.

- IT IS NOT OF THE FORM x^2+a^2 SO
  NONE OF OUR PREVIOUS INTEGRALS APPLY.

KEY: COMPLETE THE SQUARE

$$x^2+2x+2 = (x+1)^2+1$$

AND MAKE THE u-SUB $u = x+1$.
$du = dx$, AND

$$2x+3 = 2(x+1)+1,$$

SO $$\int \frac{2x+3}{x^2+2x+2}\,dx = \int \frac{2u+1}{u^2+1}\,du$$

AND WE CAN DO THIS!

$$\int \frac{2x+3}{x^2+2x+2}\,dx = \int \frac{2u+1}{u^2+1}\,du$$

$$= \int \frac{2u}{u^2+1}\,du + \int \frac{1}{u^2+1}\,du$$

$$= \ln(u^2+1) + \text{ARCTAN}(u) + c$$

$$= \ln(x^2+2x+2) + \text{ARCTAN}(x+1) + c.$$

COMPLETING THE SQUARE ON

$$x^2 + bx + c$$

GIVES $\left(x+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$.

IN GENERAL, THE SUBSTITUTION WILL BE

$$u = x + \frac{b}{2}.$$

EXAMPLE (COMPLETING THE SQUARE)

$$\int \frac{4x+4}{4x^2-8x+13}\,dx$$

$$= \int \frac{x+1}{x^2-2x+\frac{13}{4}}\,dx$$ (SEEING x^2 MAKES COMPLETING THE SQUARE CLEARER)

$$= \int \frac{(x-1)+2}{(x-1)^2+\frac{9}{4}}\,dx$$ (USE $u=x-1$)

$$= \int \frac{u+2}{u^2+\frac{9}{4}}\,du = \int \frac{u}{u^2+\frac{9}{4}}\,du + \int \frac{2}{u^2+\frac{9}{4}}\,du$$

$$= \frac{1}{2}\ln\left(u^2+\frac{9}{4}\right) + \frac{2}{\left(\frac{3}{2}\right)}\,\text{ARCTAN}\left(\frac{u}{\left(\frac{3}{2}\right)}\right) + c$$

$$= \frac{1}{2}\ln\left(x^2-2x+\frac{13}{4}\right) + \frac{4}{3}\,\text{ARCTAN}\left(\frac{2}{3}(x-1)\right) + c.$$

WHAT IF HAVE

$$\int \frac{p(x)}{q(x)}\,dx \quad \text{AND} \quad \deg p(x) \geq \deg q(x) \ ?$$

HAVE TO DO POLYNOMIAL DIVISION.
I WILL <u>NOT</u> ASK ABOUT THIS ON AN EXAM.

EXAMPLE (WITH ALMOST EVERYTHING THAT COULD EVER HAPPEN, BUT NOT A REPEATED QUADRATIC)

$$\int \frac{x^4+5x^3+8x^2+4x+2}{x^4+2x^3+2x^2}\,dx$$

(DON'T NEED TO KNOW) DIVISION

$$= \int 1 + \frac{3x^3+6x^2+4x+2}{x^4+2x^3+2x^2}\,dx$$

FACTOR

$$= \int 1 + \frac{3x^3+6x^2+4x+2}{x^2(x^2+2x+2)}\,dx$$

PARTIAL FRACTIONS

$$= \int 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{2x+3}{x^2+2x+2}\,dx$$

COMPLETE SQ.

$$= \int 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{2(x+1)+1}{(x+1)^2+1}\,dx$$

$$= x + \ln|x| - \frac{1}{x} + \ln\left((x+1)^2+1\right) + \text{ARCTAN}\,(x+1) + c.$$

USING KNOWN INTEGRALS AND $u=x+1$ FOR LAST ONE ← DID THIS EARLIER!

$$\frac{3x^3+6x^2+4x+2}{x^2(x^2+2x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+2}$$

$$3x^3+6x^2+4x+2 = Ax(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)x^2$$

$$3x^3+6x^2+4x+2 = (A+C)x^3 + (2A+B+D)x^2 + (2A+2B)x + 2B$$

$2B = 2$, $B=1$

$2A+2B = 4$, $A=1$

$A+C = 3$, $C=2$

$2A+B+D = 6$, $D=3$