L'HÔPITAL'S RULE

- L'HÔPITAL'S RULE IS A TOOL FOR CALCULATING LIMITS
- IT IS DESIGNED FOR INDETERMINATE FORMS

$$\frac{0}{0} \quad \text{AND} \quad \frac{\pm\infty}{\pm\infty}.$$

- BY USING TRICKS, IT CAN ALSO BE USED ON THE INDETERMINATE FORMS

$$0 \cdot (\pm\infty)$$

$$(\pm\infty) - (\pm\infty)$$

$$0^0,\ 1^{\pm\infty},\ (\pm\infty)^0.$$

L'HÔPITAL'S RULE

SUPPOSE EITHER

1) $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

OR 2) i) $\lim_{x \to a} f(x) = \pm\infty$ AND

ii) $\lim_{x \to a} g(x) = \pm\infty$.

SUPPOSE, ALSO, THAT

- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ EXISTS ($\pm\infty$ ALLOWED)

- $g'(x) \neq 0$ NEAR a.

THEN

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

WHAT'S GOING ON?

- WANT TO CACCULATE $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$.

- FIRST PART SAYS WE'LL DEAL WITH

$$\frac{0}{0} \text{ OR } \frac{\pm\infty}{\pm\infty}.$$

- LAST PART SAYS ANSWER IS

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

- BIT IN MIDDLE. FIRST, BETTER EXIST. SECOND PART, WILL NOT WORRY US SO MUCH BUT WE'LL SEE AN EXAMPLE OF WHY IT IS IMPORTANT.

EXAMPLES

1) $\lim_{x \to 2} \dfrac{4 - x^2}{\sin \pi x}$.

2) $\lim_{x \to \infty} \dfrac{x}{e^x}$

3) $\lim_{x \to \infty} \dfrac{x^2}{e^x}$

4) $\lim_{x \to \infty} \dfrac{x^n}{e^x}$

BOX THIS RESULT.
IMPORTANT : SAYS e^x GROWS FASTER THAN ANY POLYNOMIAL.

5) $a > 0$, $\lim_{x \to \infty} \dfrac{\ln x}{x^a}$

BOX THIS RESULT
IMPORTANT : SAYS $\ln x$ GROWS SLOWER THAN ANY x^a WITH $a > 0$.

6) $a > 0$, $\lim_{x \to 0+} x^a \ln x$

FIRST, $0 \cdot (-\infty)$.
HOW TO DEAL?

↳ $\boxed{\text{MUST TURN INTO } \frac{0}{0} \text{ OR } \frac{\pm\infty}{\pm\infty} \text{ IN ORDER TO APPLY L'HÔPITAL.}}$

SECOND, BOX THIS RESULT
IMPORTANT: $\ln x$ GOES TO $-\infty$ SLOWER THAN x^a GOES TO 0 (AS $x \to 0+$).

"L'HÔPITAL GOING BAD" EXAMPLES

1) $\lim_{x \to 1} \frac{x^2+1}{2x+1}$ NOT INDETERMINATE

2) $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}}$ WORKS BUT GOES IN A CIRCLE. EASIER WAYS TO SEE LIMIT IS 1.

3) $f(x) = e^{-\frac{1}{x^2}} x \sin\left(\frac{1}{x^4}\right)$

$g(x) = e^{-\frac{1}{x^2}}$

$\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ DNE

BUT $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$

BY SQUEEZE THM.

4) $f(x) = x + \cos x \sin x$
$g(x) = e^{\sin x}(x + \cos x \sin x)$

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} e^{-\sin x} = \text{DNE}$

BUT $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0$. ???

CAN CHECK THAT

$$g'(n\pi) = 2 + (-1)^n n\pi \quad \text{FOR } n \in \mathbb{Z}.$$

IF $n > 0$, EVEN, $g'(n\pi) > 0$

IF $n > 0$, ODD, $g'(n\pi) < 0$

SO $g'(x) = 0$ INFINITELY OFTEN NEAR $+\infty$.

L'HÔPITAL'S RULE – CONT'D

LAST TIME:

SUPPOSE

EITHER 1) $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

OR 2) i) $\lim_{x \to a} f(x) = \pm\infty$

ii) $\lim_{x \to a} g(x) = \pm\infty$.

SUPPOSE, ALSO, THAT

- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ EXISTS
- $g'(x) \neq 0$ NEAR a.

THEN

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

EXAMPLE: $\lim_{x\to 0+} x(\ln x)^2$

INDETERMINATE OF FORM

$$\boxed{0\cdot\infty.}$$

$$\boxed{\text{TRICK} \quad x(\ln x)^2 = \frac{(\ln x)^2}{\frac{1}{x}}.}$$

$$\lim_{x\to 0+} \frac{(\ln x)^2}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0+} \frac{2\ln x\cdot\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to 0+} -2x\ln x.$$

SAW LAST TIME $\lim_{x\to 0+} x\ln x = 0$

So $\lim_{x\to 0+} x(\ln x)^2 = 0.$

RECAP: $\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{\left(\frac{1}{x}\right)}$

$$= \lim_{x\to 0+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = 0.$$

EXAMPLE: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1, \quad \lim_{x\to\infty} x = \infty$

SO THIS IS AN INDETERMINATE OF THE FORM

$$\boxed{1^\infty.}$$

HOW TO DEAL WITH SUCH A THING:

1) LET $y(x) = \ln\left(1+\frac{1}{x}\right)^x$

2) (THE MOST WORK) CALCULATE

$$\lim_{x\to\infty} y(x).$$

3) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} y(x)}.$

$$y(x) = \ln\left(1+\frac{1}{x}\right)^x$$

$$= x \ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty} x = \infty, \quad \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right) = \ln 1 = 0$$

$\infty \cdot 0$ INDETERMINATE.

$$y(x) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$ NOW IS $\frac{0}{0}$.

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$$

So $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} y(x)} = e^1 = e.$$

EXAMPLE : $\lim_{x \to 0+} x^x$.

$\boxed{\begin{array}{l} 0^0 \text{ INDETERMINATE.} \\ \text{DO SAME AS LAST EXAMPLE.} \end{array}}$

$$y(x) = \ln x^x = x \ln x$$

$$\lim_{x \to 0+} y(x) = \lim_{x \to 0+} x \ln x = 0$$

LAST TIME.

So $\lim_{x \to 0+} x^x$

$$= e^{\lim_{x \to 0+} y(x)}$$

$$= e^0 = 1.$$

EXAMPLE : $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

INDETERMINATE OF FORM

$\frac{1}{0} - \frac{1}{0}$.

JUST DO FRACTIONS

$$\frac{1}{\sin x} - \frac{1}{x}$$

$$= \frac{x - \sin x}{x \sin x}$$ NOW IS $\frac{0}{0}$.

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{2\cos x - x \sin x}$$

$$= \frac{0}{2 - 0} = 0.$$