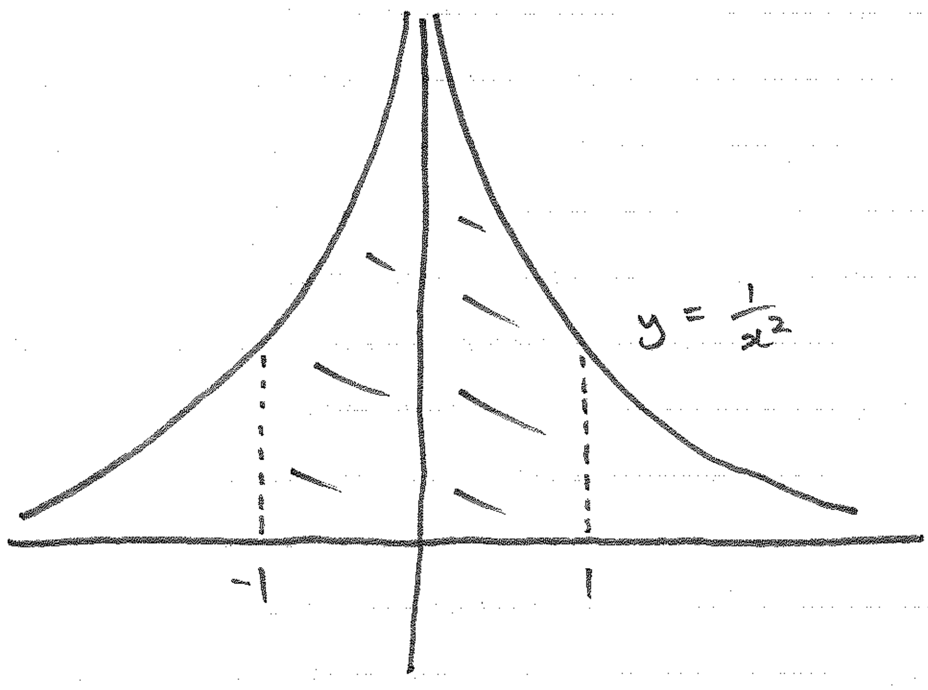
IMPROPER INTEGRALS

EXAMPLE (OF COMPLETE NONSENSE)

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \left[\frac{-1}{x} \right]_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2 .$$

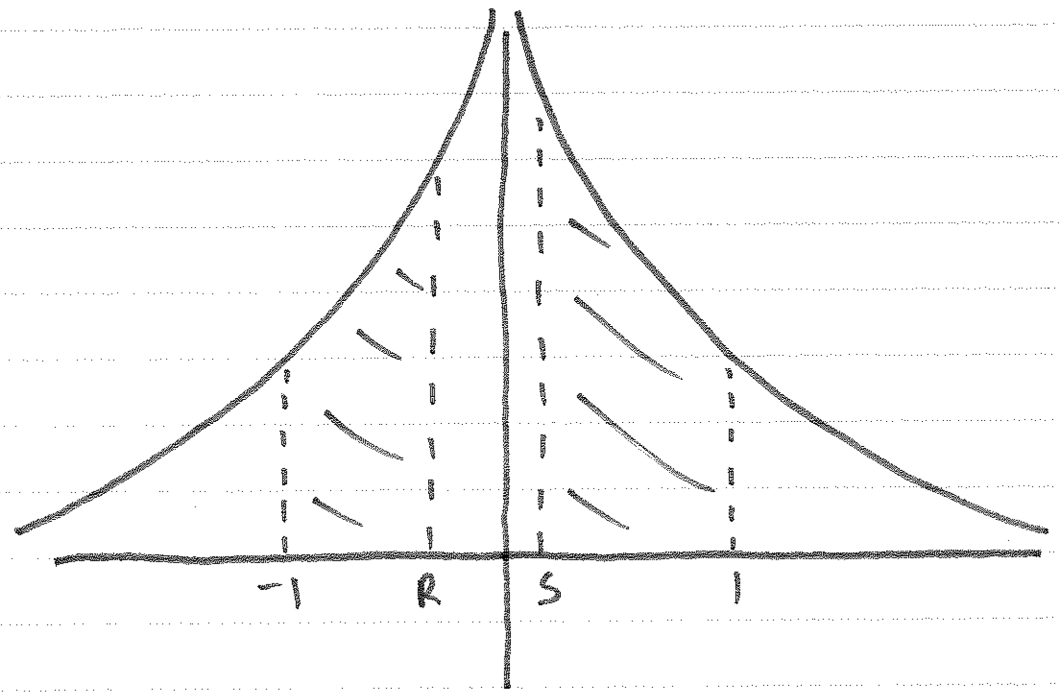
DOES THIS MAKE SENSE ?



SURELY AREA SHOULD BE POSITIVE !

SOMETHING HAS GONE BADLY WRONG.

HMM, MAYBE INTEGRATING PAST THE DISCONTINUITY OF $\frac{1}{x^2}$ AT 0 WAS A BAD IDEA.



IDEA: IF $-1 < R < 0 < S < 1$, THEN WE CAN APPROXIMATE THE AREA UNDER THE CURVE BY

$$\int_{-1}^{R} \frac{1}{x^2}\,dx + \int_{S}^{1} \frac{1}{x^2}\,dx$$

$$= \left[\frac{-1}{x}\right]_{-1}^{R} + \left[\frac{-1}{x}\right]_{S}^{1} = \left[\frac{-1}{R} - 1\right] + \left[\frac{1}{S} - 1\right].$$

HOW TO GET BETTER APPROXIMATION ?
TAKE LIMITS.

- LET R CREEP UP ON 0 FROM THE LEFT.

- LET S CREEP UP ON 0 FROM THE RIGHT.

$$\lim_{R \to 0-} \int_{-1}^{R} \frac{1}{x^2}\,dx + \lim_{S \to 0+} \int_{S}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{R \to 0-} \left[\frac{-1}{R} - 1 \right] + \lim_{S \to 0+} \left[\frac{1}{S} - 1 \right]$$

$$\text{"} = +\infty + \infty \text{"}.$$

CONCLUSION : THE AREA WASN'T -2, IT WAS INFINITE.

DEFINITION:

1) IF $f(x)$ IS CONTINUOUS ON $a \le x < b$ WITH A DISCONTINUITY AT $x = b$,

$$\int_a^b f(x)\,dx \overset{\text{DEFN}}{=} \lim_{R \to b-} \int_a^R f(x)\,dx.$$

THIS IS CALLED AN IMPROPER INTEGRAL.

THE IMPROPER INTEGRAL
- CONVERGES IF THE LIMIT IS A FINITE NUMBER
- DIVERGES IF IT DOESN'T CONVERGE.

2) IF $f(x)$ IS CONTINUOUS ON $a < x \le b$ WITH A DISCONTINUITY AT $x = a$,

$$\int_a^b f(x)\,dx \overset{\text{DEFN}}{=} \lim_{S \to a+} \int_S^b f(x)\,dx.$$

3) IF $f(x)$ IS CONTINUOUS ON $a < x < c$, AND $c < x < b$, WITH A DISCONTINUITY AT $x = c$,

$$\int_a^b f(x)\,dx \overset{\text{DEFN}}{=} \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

SOME THINGS ABOUT 3):

- $\int_a^b f(x)\,dx$ CONVERGES ONLY WHEN

  <u>BOTH</u> $\int_a^c f(x)\,dx$ AND $\int_c^b f(x)\,dx$

  CONVERGE;

  AS SOON AS EITHER

  $$\int_a^c f(x)\,dx \quad \text{OR} \quad \int_c^b f(x)\,dx$$

  DIVERGES, SO DOES

  $$\int_a^b f(x)\,dx.$$

- IT DEPENDS ON 1) AND 2).
  UNPACKING GIVES

$$\int_a^b f(x)\,dx = \lim_{R \to c-} \int_a^R f(x)\,dx + \lim_{S \to c+} \int_S^b f(x)\,dx.$$

EXAMPLES:

1) $\int_0^1 \frac{1}{x^2}\,dx$ DIVERGES

$$\lim_{s\to 0+} \int_s^1 \frac{1}{x^2}\,dx = \lim_{s\to 0+} \left[\frac{1}{s} - 1\right] = +\infty.$$

2) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$= \lim_{s\to 0+} \int_s^1 \frac{1}{\sqrt{x}}\,dx = \lim_{s\to 0+} \Big[2\sqrt{x}\Big]_s^1$$

$$= \lim_{s\to 0+} \left[2 - 2\sqrt{s}\right] = 2. \quad \text{CONVERGES}$$

OMITTED THIS → 3) $\int_0^1 \frac{1}{x}\,dx$

$$= \lim_{s\to 0+} \int_s^1 \frac{1}{x}\,dx = \lim_{s\to 0+} \Big[\ln x\Big]_s^1$$

$$= \lim_{s\to 0+} \left(-\ln s\right) = +\infty. \quad \text{DIVERGES}$$

4) $$\int_0^e \ln x \, dx$$

$$= \lim_{s \to 0+} \int_s^e \ln x \, dx$$

INTEGRATION BY PARTS GIVES

$$\int_s^e \ln x \, dx = \Big[x \ln x - x \Big]_s^e$$
$$= e - e - s\ln s + s$$
$$= s \ln s.$$

So $$\int_0^e \ln x \, dx = \lim_{s \to 0+} s \ln s$$

$$= 0$$ USING L'HOPITAL TRICKERY.

5) $$\int_0^2 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx$$

$$= \int_0^1 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx + \int_1^2 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx$$

$$= \lim_{R \to 1-} \int_0^R \frac{1}{(x-1)^{\frac{1}{3}}}\,dx + \lim_{S \to 1+} \int_S^2 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx$$

$$= \lim_{R \to 1-} \left[\frac{(x-1)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)} \right]_0^R + \lim_{S \to 1+} \left[\frac{(x-1)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)} \right]_S^2$$

$$= \lim_{R \to 1-} \left[\frac{(R-1)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)} - \frac{1}{\left(\frac{2}{3}\right)} \right] + \lim_{S \to 1+} \left[\frac{1}{\left(\frac{2}{3}\right)} - \frac{(S-1)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)} \right]$$

$$= 0 - \frac{1}{\left(\frac{2}{3}\right)} + \frac{1}{\left(\frac{2}{3}\right)} - 0 = 0.$$

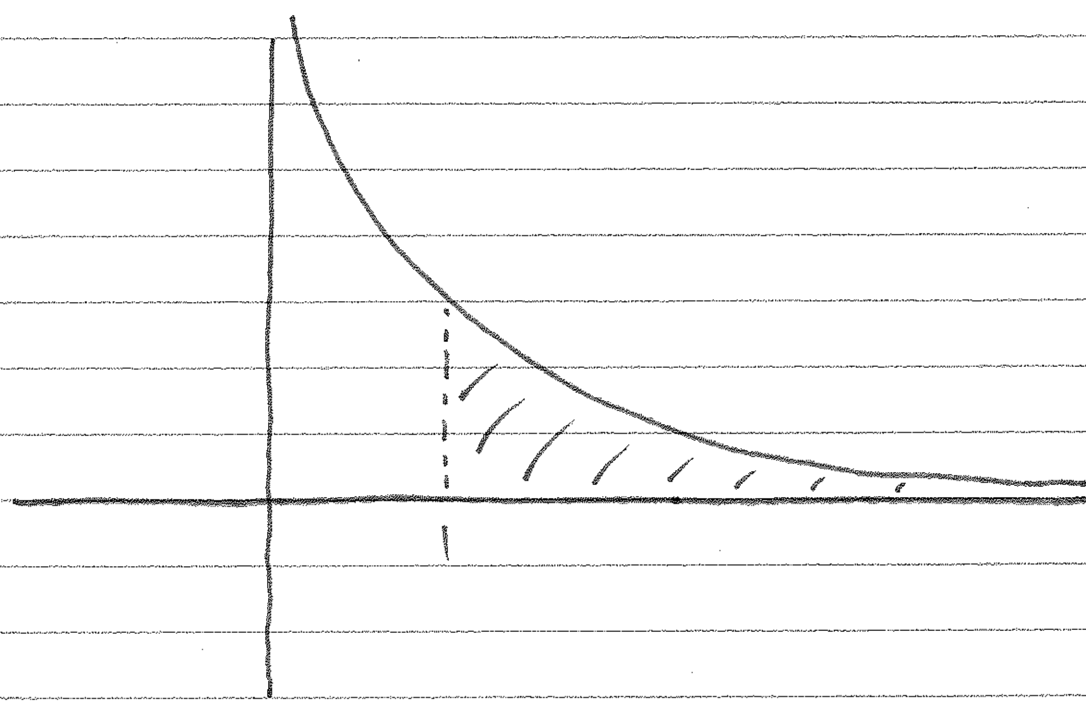
IMPROPER INTEGRALS II

LAST TIME: WE DEALT WITH DISCONTINUOUS INTEGRANDS

e.g. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx.$

TODAY: LIMITS WHICH ARE $\pm\infty$.

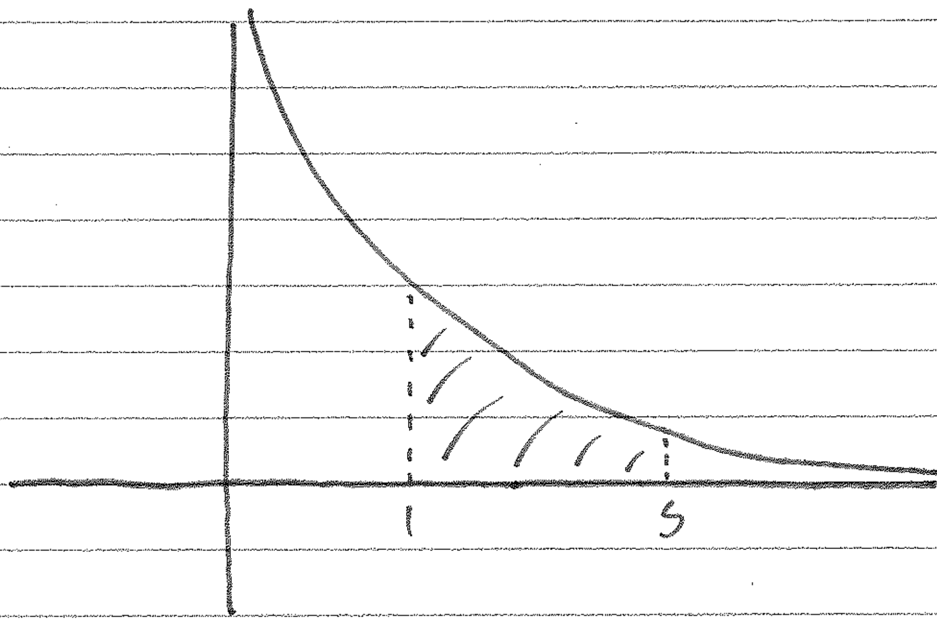
EXAMPLE: $\displaystyle\int_{1}^{\infty} \frac{1}{x^2}\,dx$



IF $S > 1$

$$\int_1^S \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^S = \frac{-1}{S} - \frac{-1}{1} = 1 - \frac{1}{S}$$

GIVES THE AREA



WE GET CLOSER TO THE AREA WE WANT
BY LETTING S GET BIGGER

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{S\to\infty}\int_1^S \frac{1}{x^2}\,dx = \lim_{S\to\infty}\left(1 - \frac{1}{S}\right) = 1.$$

DEFINITION :

1) IF $f(x)$ IS CONTINUOUS ON $x \leq a$

$$\int_{-\infty}^{a} f(x)\,dx \overset{\text{DEFN}}{=} \lim_{R \to -\infty} \int_{R}^{a} f(x)\,dx$$

THIS IS ANOTHER IMPROPER INTEGRAL.

THE IMPROPER INTEGRAL
- CONVERGES IF THE LIMIT IS A FINITE NUMBER
- DIVERGES IF IT DOESN'T CONVERGE.

2) IF $f(x)$ IS CONTINUOUS ON $x \geq a$

$$\int_{a}^{\infty} f(x)\,dx \overset{\text{DEFN}}{=} \lim_{S \to \infty} \int_{a}^{S} f(x)\,dx$$

3) IF $f(x)$ IS CONTINUOUS

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

$$= \lim_{R \to -\infty} \int_{R}^{a} f(x)\,dx + \lim_{S \to \infty} \int_{a}^{S} f(x)\,dx .$$

EXAMPLES:

1) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ CONVERGES

AND IS EQUAL TO 1.

2) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$

$$= \lim_{s\to\infty} \int_2^s \frac{1}{\sqrt{x}}\,dx = \lim_{s\to\infty} \Big[2\sqrt{x}\Big]_2^s$$

$$= \lim_{s\to\infty} \left(2\sqrt{s} - 2\right) = \infty \quad \text{DIVERGES.}$$

OMITTED THIS. → 3) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

$$= \lim_{s\to\infty} \ln s = \infty \quad \text{DIVERGES}$$

4) $\int_{-\infty}^{1} e^{x}\,dx$

$$= \lim_{R\to-\infty} \int_{R}^{1} e^{x}\,dx = \lim_{R\to-\infty} \left[e^{x} \right]_{R}^{1}$$

$$= \lim_{R\to-\infty} \left(e - e^{R} \right) = e. \quad \text{CONVERGES}$$

(BONUS: RELATE THIS CALCULATION TO THE ONE FROM LAST TIME

$$\int_{0}^{e} \ln x\,dx = 0 \;)$$

5) $\displaystyle\int_{-\infty}^{0} x e^{x}\, dx$

INTEGRATION BY PARTS GIVES

$$\int_{R}^{0} x e^{x}\, dx = (1-R)e^{R} - 1.$$

L'HÔPITAL GIVES

$$\lim_{R\to-\infty} (1-R)e^{R} \overset{\text{TRICK}}{=} \lim_{R\to-\infty} \frac{1-R}{e^{-R}} \overset{\text{L'H}}{=} \lim_{R\to-\infty} \frac{-1}{-e^{-R}} = 0.$$

$$\text{So } \int_{-\infty}^{0} x e^{x}\, dx = \lim_{R\to-\infty} \int_{R}^{0} x e^{x}\, dx$$

$$= \lim_{R\to-\infty} \left[(1-R)e^{R} - 1\right] = -1.$$

CONVERGES.

6) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

$= \int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{\infty} \frac{1}{1+x^2}\, dx$

$= \lim_{R \to -\infty} \int_{R}^{0} \frac{1}{1+x^2}\, dx + \lim_{S \to \infty} \int_{0}^{S} \frac{1}{1+x^2}\, dx$

$= \lim_{R \to -\infty} \left(-\text{ARCTAN}(R)\right) + \lim_{S \to \infty} \text{ARCTAN}(S)$

$= -\left(\frac{-\pi}{2}\right) + \frac{\pi}{2} = \pi$

CONVERGES

THEOREM (p-TEST)

- $\int_0^1 \frac{1}{x}\,dx$ AND $\int_1^\infty \frac{1}{x}\,dx$ DIVERGE.

- IF $p<1$

  $\int_0^1 \frac{1}{x^p}\,dx$ CONVERGES, $\int_1^\infty \frac{1}{x^p}\,dx$ DIVERGES.

- IF $p>1$,

  $\int_0^1 \frac{1}{x^p}\,dx$ DIVERGES, $\int_1^\infty \frac{1}{x^p}\,dx$ CONVERGES.

PROOF: A GOOD EXERCISE FOR YOU.
DO IT!!

INTUITION: I'LL SAY THINGS.