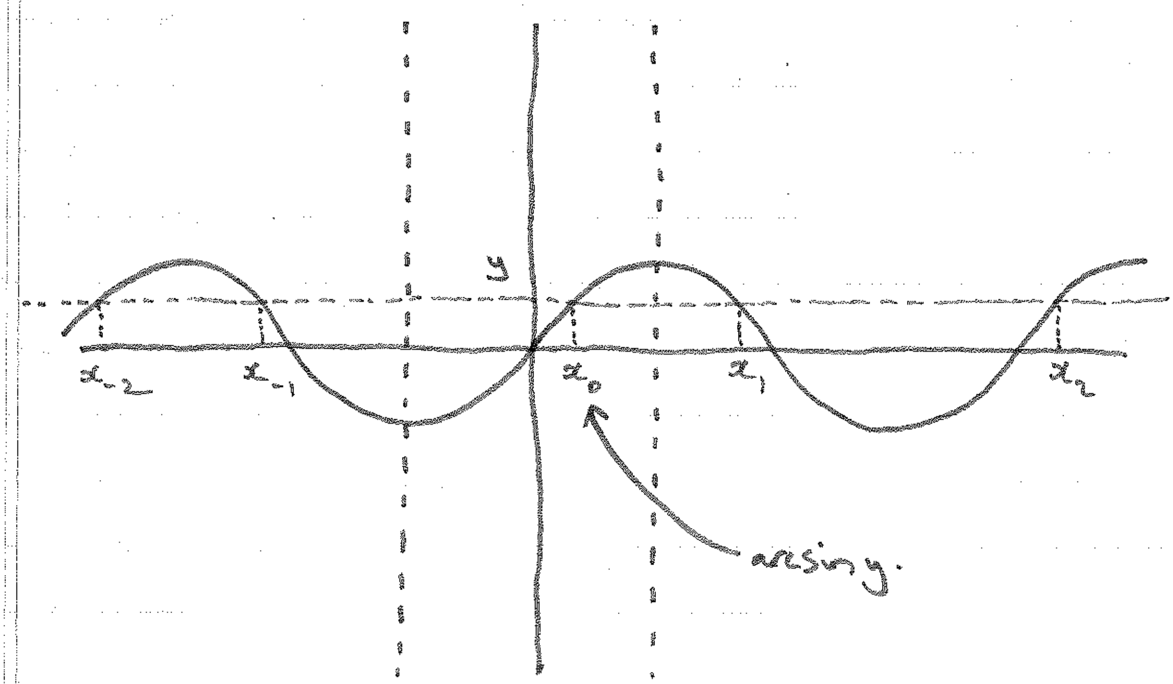
1 ARCSIN

arcsin TRIES TO UNDO sin.

PROBLEM: SUPPOSE $-1 \le y \le 1$
AND WE WANT TO FIND AN x
WITH $\sin x = y$.

SOLUTION: $x = \arcsin y$ IS ONE SUCH x.

HOW TO THINK ABOUT THIS GRAPHICALLY?



GIVEN y, THERE ARE ACTUALLY LOTS OF xs WHICH WORK.

WHICH ONE IS $\arcsin y$?

WE ALWAYS TAKE THE ONE BETWEEN

$$-\frac{\pi}{2} \text{ AND } \frac{\pi}{2}$$

TO DEFINE ARCSIN.

arcsin y IS DEFINED FOR y WITH $-1 \leq y \leq 1$,
WE HAVE $-\frac{\pi}{2} \leq \arcsin y \leq \frac{\pi}{2}$,

$$\arcsin(\sin x) = x \quad \text{WHEN} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\sin(\arcsin y) = y \quad \text{WHEN} \quad -1 \leq y \leq 1.$$

EXAMPLES

1) WHAT IS $\arcsin\left(\frac{1}{2}\right)$?

SINCE $-\frac{\pi}{2} \le \frac{\pi}{6} \le \frac{\pi}{2}$

WE HAVE

$$\arcsin\left(\frac{1}{2}\right) = \arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}.$$

2) FIND ALL x SUCH THAT

$$\sin x = \frac{\sqrt{3}}{2}.$$

ONE x IS GIVEN BY $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

LOOK AT GRAPH:

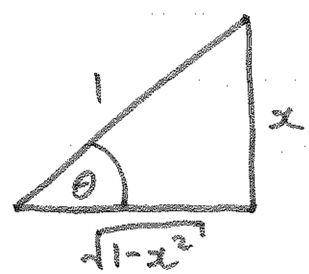
$$2n\pi + \frac{\pi}{3}, \quad (2n+1)\pi - \frac{\pi}{3}$$

AS n GOES THROUGH THE INTEGERS.

3) $\cos(\arcsin(x))$?

IF $\theta = \arcsin x$

THEN $\sin\theta = \sin(\arcsin x) = x.$



AND SO

$$\cos(\arcsin x) = \cos\theta = \sqrt{1-x^2}$$

$$\frac{d}{dx}\left(\arcsin x\right) = ?$$

SUPPOSE $y = \arcsin x$

THEN $\sin y = \sin(\arcsin x) = x$

$$(\cos y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin x)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c}$$

EXAMPLE

$$\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\, dx$$

WANT IT TO LOOK LIKE THE FRIEND WE JUST MADE.

PROBLEM: HAVE $4x^2$.
WISH WE HAD x^2.

CAN ALMOST DO THIS.
JUST GOT TO USE u.

WANT

$$u^2 = 4x^2$$

SO LET

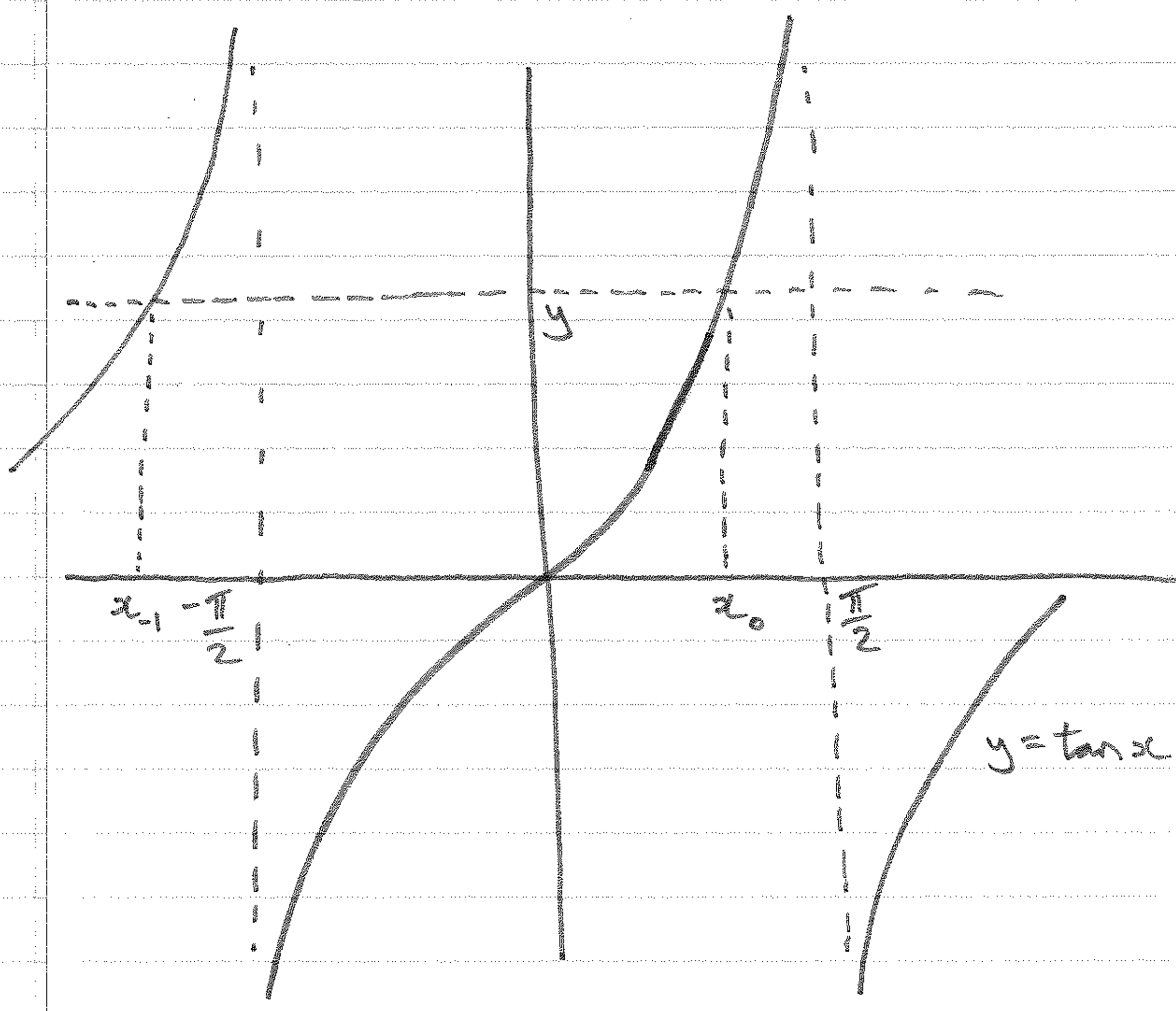
$$u = 2x, \qquad du = 2\,dx$$

$$\tfrac{1}{2}\, du = dx$$

$$\frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\, du = \frac{1}{2}\Big[\arcsin u\Big]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - 0\right] = \frac{\pi}{12}.$$

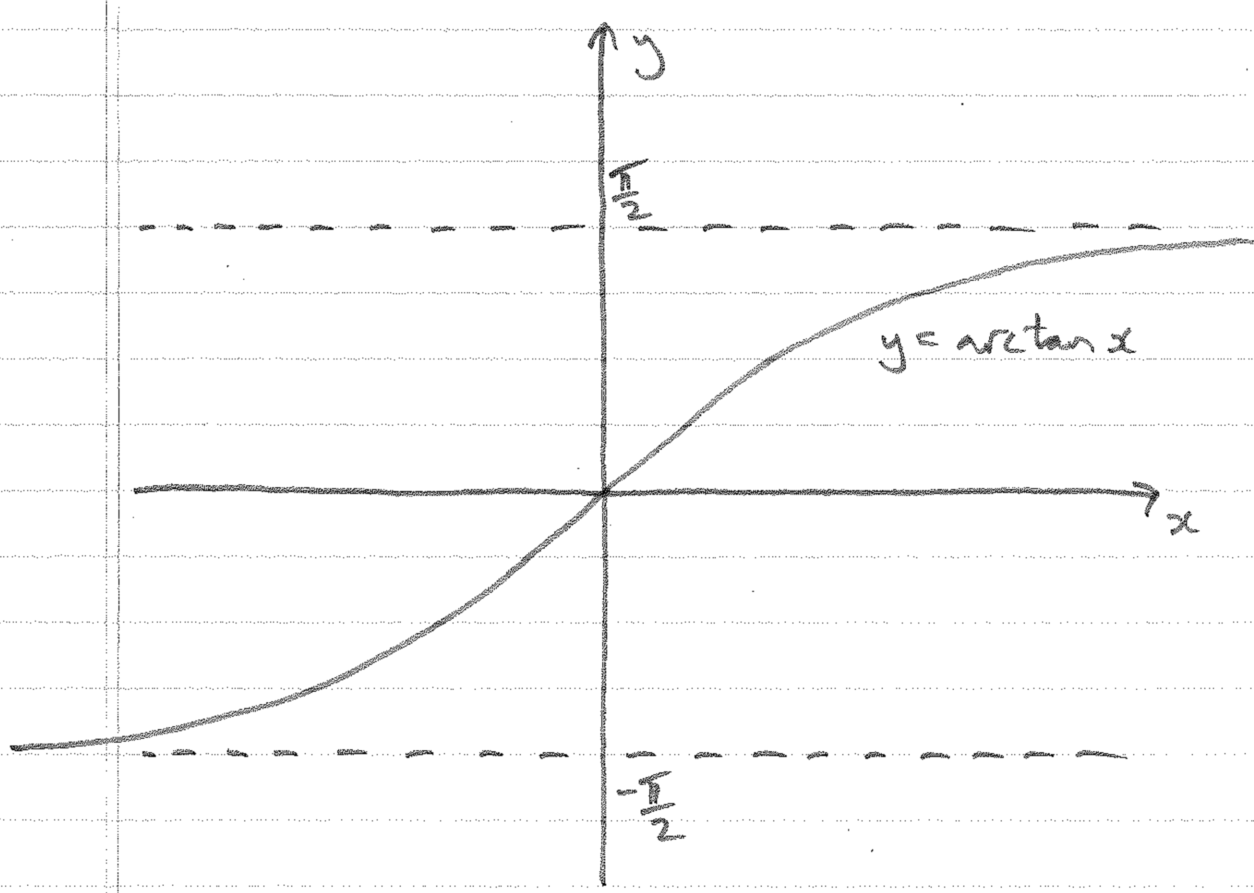
2 ARCTAN



TO DEFINE $\arctan y$ WE TAKE THE x BETWEEN $-\frac{\pi}{2}$ AND $\frac{\pi}{2}$.

$\arctan y$ IS DEFINED FOR ALL y,
WE HAVE $-\frac{\pi}{2} < \arctan y < \frac{\pi}{2}$,

$$\arctan(\tan x) = x \quad \text{WHEN } -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$\tan(\arctan y) = y \quad \text{FOR ALL } y.$$



$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \arctan x = \frac{-\pi}{2}.$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + c$$

EXAMPLE (IMPORTANT)

$$\int \frac{1}{a^2+x^2}\,dx$$

WISH a^2+x^2 WAS $1+x^2$.
u-SUB?
ONLY REALLY HAVE CONTROL OVER
THE x^2 BIT...

JUST AS GOOD:
WISH a^2+x^2 WAS $a^2(1+x^2) = a^2+a^2x^2$

WANT

$$a^2+x^2 = a^2+a^2u^2$$

SO LET

$$x = au$$
$$dx = a\,du$$

$$\int \frac{a}{a^2+a^2u^2}\,du = \frac{1}{a}\int \frac{1}{1+u^2}\,du$$

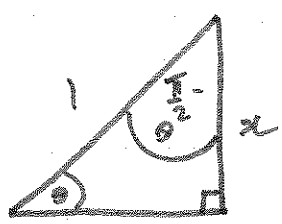
$$= \frac{1}{a}\arctan u + c$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c.$$

$$\boxed{\int \frac{1}{a^2+x^2}\,dx = \arctan\left(\frac{x}{a}\right) + c}$$

3 ARCCOS

$$\cos\left(\frac{\pi}{2} - \arcsin(x)\right) = \cos\left(\frac{\pi}{2} - \theta\right) = x$$



$\theta = \arcsin x$

$\sin\theta = x$

$$\boxed{\arccos y = \frac{\pi}{2} - \arcsin y}$$