Math 31A
Differential and Integral Calculus

Midterm 2 Practice

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of ?? points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ______________________

Discussion: ___________________________

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
Total:	0	

Problem 1.

A police car traveling west towards UCLA at a constant speed of 160km/h is in pursuit of a truck traveling north from UCLA at a constant speed of 140km/h.

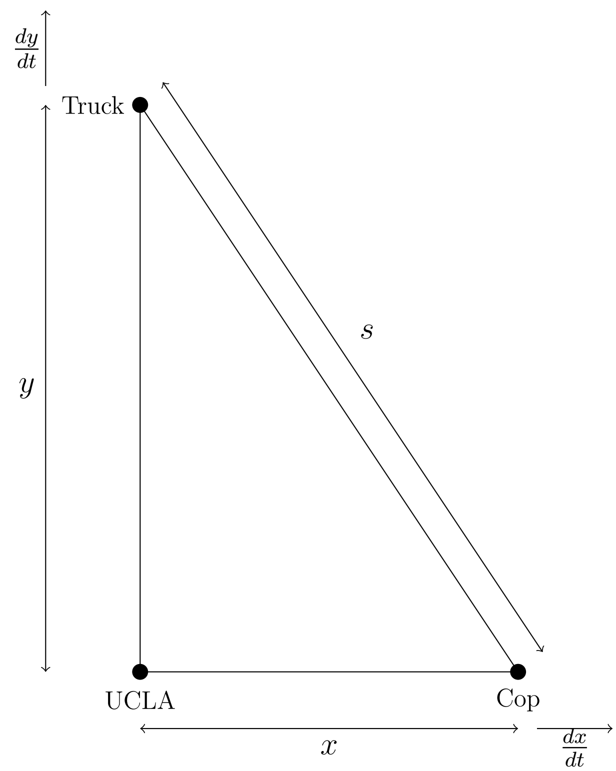
At time $t = 0$, the police car is 20km east and the truck is 30km north of UCLA.

Calculate the rate at which the distance between the vehicles is changing:

(a) at time $t = 0$;

(b) 5 minutes later.

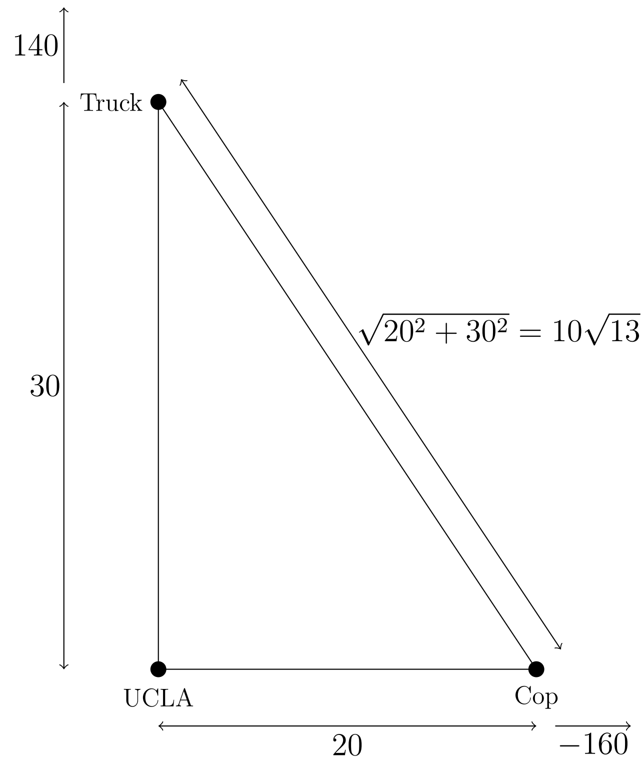
Solution: First, I'd draw a picture of the movie.

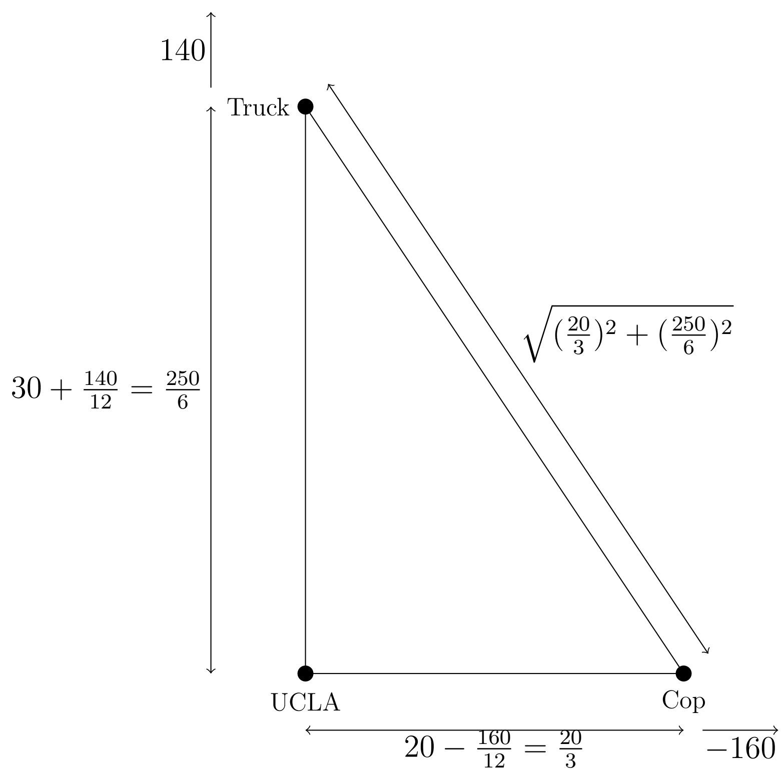


We see that $x^2 + y^2 = s^2$ so $x\frac{dx}{dt} + y\frac{dy}{dt} = s\frac{ds}{dt}$.

Then I'd draw pictures of the frames.

The $t = 0$ and $t = \frac{1}{12}$ frames look as follows.





(a) We find $20 \cdot (-160) + 30 \cdot (140) = 10\sqrt{13} \cdot \frac{ds}{dt}$.

So $\frac{ds}{dt} = \frac{2\cdot(-160)+3\cdot(140)}{\sqrt{13}} = \frac{100}{\sqrt{13}}$.

(b) We find $\frac{20}{3} \cdot (-160) + \frac{250}{6} \cdot (140) = \sqrt{(\frac{20}{3})^2 + (\frac{250}{6})^2} \cdot \frac{ds}{dt}$.

So $\frac{ds}{dt} = \frac{\frac{20}{3}\cdot(-160)+\frac{250}{6}\cdot(140)}{\sqrt{(\frac{20}{3})^2+(\frac{250}{6})^2}} = \frac{2860}{\sqrt{641}}$.

The numbers would never be so horrible in the real midterm and I wouldn't care so much about getting an actual number; the first expression would be fine.

Problem 2.

A cone has volume $V = \frac{\pi}{3}r^2h$ and surface area $S = \pi r\sqrt{r^2+h^2}$. Find the dimensions of the cone with surface area 1 and maximal volume.

Solution: We should turn the problem into its generic form.

By this I mean, we want to write the problem as:

maximize/minimize BLAH subject to the constraint BLAH.

The relevant sentence is

$$\text{maximize } \frac{\pi}{3}r^2h \text{ subject to the constraint } \pi r\sqrt{r^2+h^2} = 1.$$

An equivalent problem is

$$\text{maximize } r^4h^2 \text{ subject to the constraint } \pi^2r^2(r^2+h^2) = 1.$$

I'm just trying to make my life a little easier here. I've done the following:

- Got rid of the positive constant $\frac{\pi}{3}$ to get r^2h. That won't make a difference to maximizing the function.
- Squared r^2h to get r^4h^2. That also won't make a difference to maximizing the function.
- Squared both sides of the constraint.

If $\pi^2r^2(r^2+h^2) = 1$, then $(r^2+h^2) = \frac{1}{\pi^2r^2}$, so that $h^2 = \frac{1}{\pi^2r^2} - r^2$.

The problem becomes

$$\text{maximize } T(r) = r^4\left(\frac{1}{\pi^2r^2} - r^2\right) = \frac{r^2}{\pi^2} - r^6.$$

$T'(r) = \frac{2r}{\pi^2} - 6r^5$. So $T'(r) = 0$ when $\frac{2r}{\pi^2} = 6r^5$. Since $r = 0$ does not make sense for the physical problem, this gives $\frac{2}{\pi^2} = 6r^4$, so $r = \frac{1}{\sqrt[4]{3\pi^2}}$.

This is a maximum since $T''(r) = \frac{2}{\pi^2} - 30r^4$ and so $T''(\frac{1}{\sqrt[4]{3\pi^2}}) = \frac{2}{\pi^2} - \frac{30}{3\pi^2} < 0$.

Finally, $h = \sqrt{\frac{1}{\pi^2r^2} - r^2} = \sqrt{\frac{\sqrt{3}\pi}{\pi^2} - \frac{1}{\sqrt{3}\pi}} = \sqrt{\frac{\sqrt{3}}{\pi} - \frac{1}{\sqrt{3}\pi}} = \frac{\sqrt{\sqrt{3}-\frac{1}{\sqrt{3}}}}{\sqrt{\pi}}$.

Problem 3.

Suppose $f(x) = (x^3 - 3x)^{\frac{1}{3}}$. It is then the case that

$$f'(x) = (x^2 - 1)(x^3 - 3x)^{-\frac{2}{3}} \quad \text{and} \quad f''(x) = -2(x^2 + 1)(x^3 - 3x)^{-\frac{5}{3}}.$$

(a) What are the critical points of $f(x)$?

(b) On what intervals is $f(x)$ increasing/decreasing?

(c) What are the local maxima and minima of $f(x)$?

(d) Describe the concavity of $f(x)$ on the relevant intervals?

(e) Does $f(x)$ have inflection points?

(f) What are $\lim_{x\to\infty} \frac{f(x)}{x}$ and $\lim_{x\to-\infty} \frac{f(x)}{x}$?

(g) Sketch $y = f(x)$.

Solution:

(a) $f'(x) = 0$ when $x = -1$ or $x = 1$. $f'(x)$ does not exist when $x = -\sqrt{3}$, $x = 0$ or $x = \sqrt{3}$. The "does not exist" is a result of division by zero, so we expect something like an infinite slope at these points, as opposed to a spike.

(b) $f(x)$ is increasing on $(-\infty, -1)$ and $(1, \infty)$. $f(x)$ is decreasing on $(-1, 1)$.

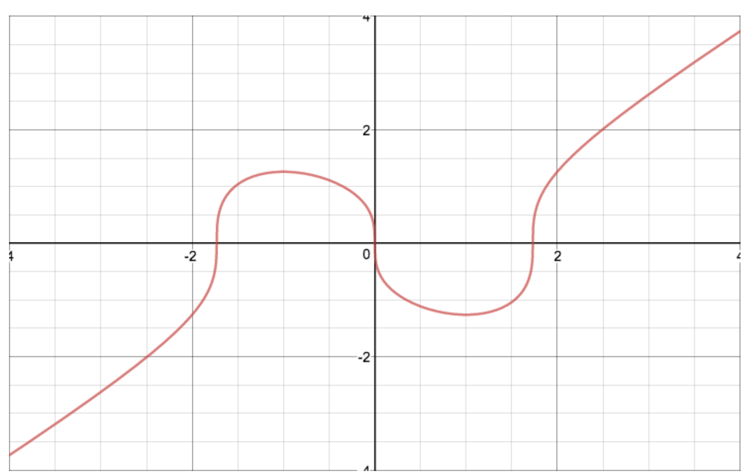
(c) There's a local maximum at $x = -1$ and a local minimum at $x = 1$.

(d) $f(x)$ is concave up on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.
$f(x)$ is concave down on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

(e) There are inflection points at $x = -\sqrt{3}$, $x = 0$ and $x = \sqrt{3}$.

(f) They are both 1.

(g)



Graph of $y = (x^3 - 3x)^{\frac{1}{3}}$.

Problem 4.

Let $f(x) = (\sin x)^2 + \cos x$.

x	$\cos x$	$\sin x$
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
π	-1	0

(a) Find the critical points of $f(x)$ on $-\frac{\pi}{6} < x < \frac{7\pi}{6}$.

Solution:

$$f'(x) = 2 \sin x \cos x - \sin x = (2 \cos x - 1) \sin x,$$

so $f'(x)$ is always defined and $f'(x) = 0$ when $\cos x = \frac{1}{2}$ or $\sin x = 0$.

$\cos x = \frac{1}{2}$ when $x = \frac{\pi}{3}$, and $\sin x = 0$ when $x = 0$ or π.

The critical points are at $x = 0$, $\frac{\pi}{3}$ and π.

(b) Classify the critical points found in a),

i.e. say whether each is local maximum, local minimum or neither.

You do not need to give the y-values; the x-values will do.

Note: the second derivative test does not fail, so you could try and use it.

Solution:

$$f''(x) = (2 \cos x - 1) \cos x - 2(\sin x)^2$$

so $f''(0) = (2-1) \cdot 1 - 2 \cdot 0^2 = 1 > 0$ and 0 is a local minimum.

$f''(\frac{\pi}{3}) = 0 - 2(\neq 0)^2 < 0$, so $\frac{\pi}{3}$ is a local maximum.

$f''(\pi) = (-2-1) \cdot (-1) - 2 \cdot 0^2 = 3 > 0$, so π is a local minimum.

(c) Say where $f(x)$ is increasing and decreasing on $0 \leq x \leq \pi$.
No justification is necessary. You'll get full credit for the correct answer.

Solution: It is increasing on $0 < x < \frac{\pi}{3}$ and decreasing on $\pi/3 < x < \pi$.

(d) Find the x-values where $f(x)$ attains its maximum and minimum on $0 \leq x \leq \frac{11\pi}{12}$.
It is not necessary to give the y-values; the x-values will do.
Hint: it might be helpful to calculate $f(\frac{\pi}{2})$ and make use of part c).

Solution: The candidate points are $x = 0$, $\frac{\pi}{3}$ and $\frac{11\pi}{12}$.
$f(0) = 1$.
$f(\frac{\pi}{3}) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$.
$f(\frac{\pi}{2}) = 1$.
Since f is decreasing on $\pi/3 < x < \frac{11\pi}{12}$ we see that $\frac{11\pi}{12}$ is the global minimum.
$\frac{\pi}{3}$ is the global maximum.

(e) Justify the existence of an inflection point between 0 and $\frac{\pi}{2}$.
Warning: don't try to find the x-value or y-value.
If you think you have found it/them then you are either a genius or incorrect.

Solution: $f''(0) > 0 > f''(\frac{\pi}{3})$.
$f''(x)$ is continuous so there is an x_0 with

$$0 < x_0 < \frac{\pi}{3} < \frac{\pi}{2} \text{ and } f''(x_0) = 0$$

at which the concavity changes.
Alternatively, a picture of $\cup$ and $\cap$ joined together works!