Math 31A
Differential and Integral Calculus

Midterm 2

Instructions: You have 50 minutes to complete this exam. There are three questions, worth a total of 42 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

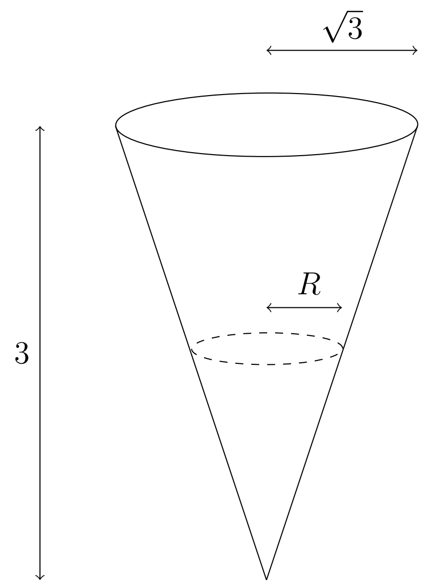
Student ID number: ________________________

Discussion: ___________________________

Question	Points	Score
1	14	
2	12	
3	16	
Total:	42	

Problem 1.

Recall that the volume of a circular cone is given by $\frac{\pi}{3}r^2h$, and its surface area is given by $\pi r\sqrt{r^2+h^2}$. In each of these equations, r is the radius of the circle at the top of the cone, and h is its height.



Consider the cone drawn above. Its radius is $\sqrt{3}$ meters and its height is 3 meters.

The cone is filled with water.

Due to holes in the cone (which are not drawn), water leaves the cone at a rate proportional to the surface area of the cone in contact with the water; the formula is

$$\frac{dV}{dt} = -\frac{\sqrt{3}}{2}A$$

where V is the volume of water, A is the surface area of the cone in contact with the water, and the units for time are hours.

Answer the questions over the page. You may want to tear out this page to look at. If you do so, it is sensible NOT to write valuable work on this page.

(a) [6pts.] Let R be as in the picture. Write formulae for V and A in terms of R.

(b) [3pts.] Differentiate the formula for V to find $\frac{dV}{dt}$.

(c) [3pts.] With R as in the picture, what is $\frac{dR}{dt}$ when $R = 1$ meter?

(d) [2pts.] Calculate the rate at which the water level changes when it is 1 meter high.

Solution:

(a) Let R be as in the picture, and h be the height of the water.
Then $\frac{h}{R} = \frac{3}{\sqrt{3}} = \sqrt{3}$, so $h = R\sqrt{3}$.
We find $V = \frac{\pi}{3}R^2(R\sqrt{3}) = \frac{\pi}{\sqrt{3}} \cdot R^3$
and $A = \pi R\sqrt{R^2 + (R\sqrt{3})^2} = \pi R\sqrt{4R^2} = 2\pi \cdot R^2$.

(b) Moreover, $\frac{dV}{dt} = \sqrt{3} \cdot \pi \cdot R^2 \cdot \frac{dR}{dt}$.

(c) From what we're told, we get

$$\sqrt{3} \cdot \pi \cdot R^2 \cdot \frac{dR}{dt} = -\frac{\sqrt{3}}{2} \cdot 2\pi \cdot R^2$$

so that $\frac{dR}{dt} = -1$.

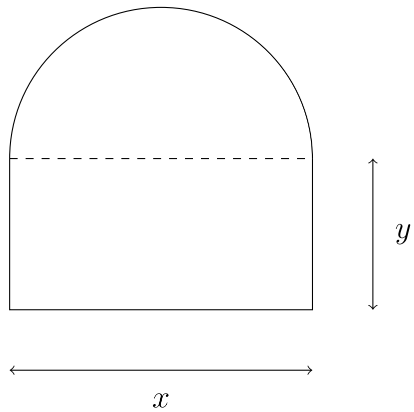
(d) $\frac{dh}{dt} = \sqrt{3} \cdot \frac{dR}{dt} = -\sqrt{3}$ meter per hour.

Problem 2.

(a) [4pts.] You wish to build an enclosed field for your horses to play in.

You have $P = (400 + 300\pi)$ meters of fencing and you decide that you will build it in the shape of a semicircle on top of a rectangle.

You want to give them the biggest possible area to run around in.



Write down the problem you have to solve in the form

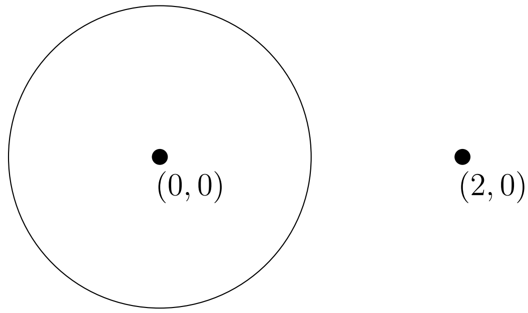
"maximize/minimize BLAH subject to the constraint BLAH."

There is no need to solve the optimization problem.

[However, a correct solution would help make up for a faulty sentence.]

Solution:
Maximize $xy + \frac{1}{2}\pi(\frac{x}{2})^2$ subject to the constraint $x + \frac{1}{2}\pi x + 2y = 400 + 300\pi$.

(b) [8pts.] Maximize $(x-2)^2 + y^2$ subject to the constraint $x^2 + y^2 = 1$.



Beware of what x and y-values make sense.

Solution: We are asking to minimize the distance (squared) from the point $(2, 0)$ to the circle with radius 1. We can rewrite the problem as

"maximize $(x^2 + y^2) - 4x + 4$ subject to the constraint $x^2 + y^2 = 1$."

Thus, we need to maximize

$$f(x) = 1 - 4x + 4 = 5 - 4x$$

for the relevant values of x.

$f'(x) = -4$ so $f(x)$ has no critical points, but is a decreasing function. It obtains its maximum at the left most x-value, that is, $x = -1$.

We see that $f(-1) = 9$ so the maximum is 9 obtained at $(x, y) = (-1, 0)$.

Drawing a circle of radius 3 around $(2, 0)$ gives a proof that uses no calculus.

Problem 3.

Let $f(x) = 9 \cdot x \cdot (x^2 - 7)^{\frac{2}{3}}$. Then

$$f'(x) = 21 \cdot (x^2 - 3) \cdot (x^2 - 7)^{-\frac{1}{3}}$$

and

$$f''(x) = 28 \cdot x \cdot (x^2 - 9) \cdot (x^2 - 7)^{-\frac{4}{3}}.$$

(a) [2pts.] What are the critical points of $f(x)$?

(b) [3pts.] On what intervals is $f(x)$ increasing/decreasing?

(c) [2pts.] What are the local maxima and minima of $f(x)$?

(d) [3pts.] On the interval $-1 \leq x \leq 2$, what are the x-values where the global maximum and minimum of $f(x)$ occur? y-values are not necessary.

(e) [2pts.] Describe the concavity of $f(x)$ on the relevant intervals?

(f) [1pts.] Does $f(x)$ have inflection points? If so, find them.

(g) [3pts.] Sketch $y = f(x)$.
There is no need to indicate the y-values of any local maxima or minima, but you should indicated x-intercepts.

Solution:

(a) $f'(x) = 0$ when $x = -\sqrt{3}$ or $x = \sqrt{3}$. $f'(x)$ DNE when $x = -\sqrt{7}$ or $x = \sqrt{7}$.

(b) $f(x)$ is increasing when $x < -\sqrt{7}$, $-\sqrt{3} < x < \sqrt{3}$, or $x > \sqrt{7}$.
$f(x)$ is decreasing when $-\sqrt{7} < x < -\sqrt{3}$ or $\sqrt{3} < x < \sqrt{7}$.

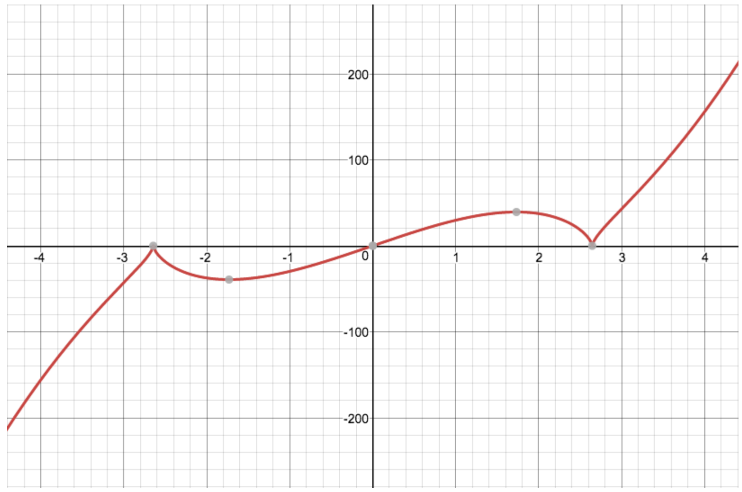
(c) $x = -\sqrt{7}$, $x = -\sqrt{3}$, $x = \sqrt{3}$, or $x = \sqrt{7}$ are max, min, max, min, respectively.

(d) Global minimum at $x = -1$. Global maximum at $x = \sqrt{3}$.

(e) $f(x)$ is concave down on $x < -3$, concave up on $-3 < x < 0$, concave down on $0 < x < 3$, and concave up on $x > 3$.

(f) Inflection points at $x = -3$, $x = 0$, and $x = 3$.

(g)



Graph of $y = 9 \cdot x \cdot (x^2 - 7)^{\frac{2}{3}}$.