Math 31A
Differential and Integral Calculus

Midterm 1 Practice

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of ?? points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ______________________

Discussion: ___________________________

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
Total:	0	

Problem 1.

Differentiate the following functions.

(a) $f(x) = (6x^4 - 3x^2 + 1)\sqrt{x}$.

(b) $f(x) = \frac{x^4 \sin x}{x^2+x+1}$

(c) $f(x) = \tan^3 x + \sin(3x)\tan(x^3)$.

(d) $f(x) = \sin(\cos(\sin x))$.

Solution:

(a) $f(x) = (24x^3 - 6x)\sqrt{x} + (6x^4 - 3x^2 + 1) \cdot \frac{1}{2\sqrt{x}}$.

(b) $f'(x) = \frac{(x^2+x+1)(4x^3 \sin x + x^4 \cos x) - x^4(\sin x)(2x+1)}{(x^2+x+1)^2}$.

(c) $f'(x) = 3\tan^2 x \sec^2 x + 3\cos(3x)\tan(x^3) + 3x^2 \sin(3x)\sec^2(x^3)$.

(d) $f'(x) = -\cos(\cos(\sin x))\sin(\sin x)\cos x$.

Problem 2.

Suppose that $y = 5 - 2x$ is the tangent line to $y = f(x)$ at $x = 3$ and that $y = 2 + 3x$ is the tangent line to $y = g(x) + x$ at $x = 3$.

What is the tangent line to the function $y = h(x) = f(x)g(x) + x^2$ at $x = 3$?

Solution: From the first tangent line, we learn that

$$f(3) = 5 - 2 \cdot 3 = -1 \quad \text{and} \quad f'(3) = -2.$$

From the second tangent line we learn that $g(3) + 3 = 2 + 3 \cdot 3$, so $g(3) = 8$, and

$$g'(3) + 1 = 3, \quad \text{so that} \quad g'(3) = 2.$$

Thus, $h(3) = f(3)g(3) + 3^2 = (-1) \cdot 8 + 9 = 1$ and

$$h'(3) = f'(3)g(3) + f(3)g'(3) + 2 \cdot 3 = (-2) \cdot 8 + (-1) \cdot 2 + 2 \cdot 3 = -12.$$

The tangent line to $y = h(x)$ at $x = 3$ is

$$y - 1 = -12(x - 3).$$

Problem 3.

(a) Find the points on the graph of $y^2 = x^3 - 3x + 1$ where the tangent line is horizontal.

Solution: Differentiating the equation of the graph implicitly gives

$$2yy' = 3x^2 - 3.$$

Setting $y' = 0$ gives $3x^2 - 3 = 0$ so that $x^2 = 1$, and either $x = -1$ or $x = 1$.

When $x = 1$, $x^3 - 3x + 1 = 1 - 3 + 1 = -1 < 0$
and so there is no corresponding y-value.

When $x = -1$, $x^3 - 3x + 1 = -1 + 3 + 1 = 3$
and we get either $y = \sqrt{3}$ or $y = -\sqrt{3}$.

There are two points: $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$.

(b) The tangent line to the graph of

$$y^2 + 2xy - y = x^3 - 2x^2 + 2x$$

at the point $(0, 1)$ crosses the graph at one other point. Find it.

Solution: Differentiating the equation of the graph implictly gives

$$2yy' + 2y + 2xy' - y' = 3x^2 - 4x + 2$$

and so

$$y' = \frac{3x^2 - 4x - 2y + 2}{2x + 2y - 1}.$$

When $(x, y) = (0, 1)$ we get $y' = 0$.
This tells us that the tangent line to the graph at $(0, 1)$ is $y = 1$ which intersects the graph where

$$1 + 2x - 1 = x^3 - 2x^2 + 2x, \quad \text{i.e.} \quad x^3 - 2x^2 = x^2(x - 2) = 0.$$

We see that the other point is $(2, 1)$.

Problem 4.

Evaluate the following limits.

(a) $\lim_{x\to 0}\left[(1-\cos x)\sin\left(\frac{1}{x}\right)\right]$

Solution: Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1$,
we obtain $-(1-\cos x) \le (1-\cos x)\sin\left(\frac{1}{x}\right) \le 1-\cos x$ near $x=0$.

$\lim_{x\to 0}(1-\cos x) = 1-\cos(0) = 1-1 = 0$ and
$\lim_{x\to 0} -(1-\cos x) = -\lim_{x\to 0}(1-\cos x) = -0 = 0$.

The squeeze theorem tells us that $\lim_{x\to 0}\left[(1-\cos x)\sin\left(\frac{1}{x}\right)\right] = 0$.

(b) $\lim_{x\to 3}\frac{x-\sqrt{x+6}}{x-3}$.

Solution: We multiply by the conjugate:

$$\frac{x-\sqrt{x+6}}{x-3} = \frac{x-\sqrt{x+6}}{x-3}\cdot\frac{x+\sqrt{x+6}}{x+\sqrt{x+6}} = \frac{x^2-x-6}{(x-3)(x+\sqrt{x+6})}.$$

Then we factor:

$$\frac{x^2-x-6}{(x-3)(x+\sqrt{x+6})} = \frac{(x-3)(x+2)}{(x-3)(x+\sqrt{x+6})} = \frac{x+2}{x+\sqrt{x+6}}.$$

Since $\frac{x+2}{x+\sqrt{x+6}}$ is continuous we can plug in to obtain

$$\lim_{x\to 3}\frac{x-\sqrt{x+6}}{x-3} = \frac{3+2}{3+\sqrt{3+6}} = \frac{5}{3+3} = \frac{5}{6}.$$

(c) $\lim_{x\to 0} \frac{1-\cos x}{\sin x}$.

Solution: There are two ways I can think of.

1. Multiplying by the conjugate gives

$$\frac{1-\cos x}{\sin x} = \frac{1-\cos x}{\sin x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{(\sin x)(1+\cos x)}.$$

   Using the identity $\cos^2 x + \sin^2 x = 1$ gives

$$\frac{1-\cos^2 x}{(\sin x)(1+\cos x)} = \frac{\sin^2 x}{(\sin x)(1+\cos x)} = \frac{\sin x}{1+\cos x}.$$

   $\frac{\sin x}{1+\cos x}$ is continuous, so we can plug in to obtain

$$\lim_{x\to 0}\left[\frac{1-\cos x}{\sin x}\right] = \lim_{x\to 0}\left[\frac{\sin x}{1+\cos x}\right] = \frac{0}{1+1} = 0.$$

2. We can multiply by $1 = \frac{x}{x}$ to obtain

$$\frac{1-\cos x}{\sin x} = \frac{1-\cos x}{\sin x} \cdot \frac{x}{x} = \frac{1-\cos x}{x} \cdot \frac{x}{\sin x}.$$

   We can evaluate these limits individually:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0 \quad \text{and} \quad \lim_{x\to 0}\left(\frac{x}{\sin x}\right) = \frac{1}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)} = \frac{1}{1} = 1.$$

   So

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \lim_{x\to 0}\left(\frac{x}{\sin x}\right) = 0 \cdot 1 = 0.$$

(d) $\lim_{x\to 0+} \frac{2-\cos x}{\sin x}$.

Solution: Since $\frac{2-\cos x}{\sin x} = \frac{1}{\sin x} + \frac{1-\cos x}{\sin x}$, and $\lim_{x\to 0+} \frac{1-\cos x}{\sin x} = 0$ we expect

$$\lim_{x\to 0+}\left[\frac{2-\cos x}{\sin x}\right] = \lim_{x\to 0+}\left[\frac{1}{\sin x}\right] = \infty.$$

Thinking about what happens when we plug in a small positive number x gives a valid argument.

If x is small and positive, $2-\cos x$ is close to 1, and $\sin x$ is positive and close to 0. Thus, so $\frac{2-\cos x}{\sin x}$ is a really big positive number. Thus, $\lim_{x\to 0+} \frac{2-\cos x}{\sin x} = \infty$.