Math 31A
Differential and Integral Calculus

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are five questions, worth a total of 54 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ______________________

Discussion: ___________________________

Question	Points	Score
1	12	
2	10	
3	10	
4	20	
5	2	
Total:	54	

Problem 1.

Differentiate the following functions.

(a) [4pts.] $f(x) = \frac{x^2 \sin(x)}{1+x^2}$

(b) [4pts.] $f(x) = \sin^2(3x)\sin(4x^5)$

(c) [4pts.] $f(x) = \sqrt{1+\sqrt{1+\sqrt{1+x}}}$

Solution:

(a) $\frac{(1+x^2)[2x\sin(x)+x^2\cos(x)]-2x^3\sin(x)}{(1+x^2)^2}$

(b) $6\sin(3x)\cos(3x)\sin(4x^5) + 20x^4\sin^2(3x)\cos(4x^5)$

(c) $\frac{1}{2\sqrt{1+\sqrt{1+\sqrt{1+x}}}} \cdot \frac{1}{2\sqrt{1+\sqrt{1+x}}} \cdot \frac{1}{2\sqrt{1+x}}$.

Problem 2.

Suppose $f(x)$ and $g(x)$ are functions and you are told the following information.

- The tangent line to $y = g(x)$ at $x = 1$ is $y = 2x + 1$.
- The tangent line to $y = f(x)$ at $x = 1$ is $y = 4x + 6$.
- The tangent line to $y = f(x)$ at $x = 3$ is $y = 5x - 3$.

(a) [3pts.] What does information does the first bullet point give you?

(b) [7pts.] What is the tangent line to $y = h(x) = 4x^2 + f(g(x))$ at $x = 1$?

Solution:

(a) $g(1) = 3$ and $g'(1) = 2$.

(b) The third bullet point tells us $f(3) = 12$ and $f'(3) = 5$.

$h(1) = 4 + f(g(1)) = 4 + f(3) = 16$.

$h'(x) = 8x + f'(g(x))g'(x)$, so $h'(1) = 8 + f'(g(1))g'(1) = 8 + f'(3)g'(1) = 18$.

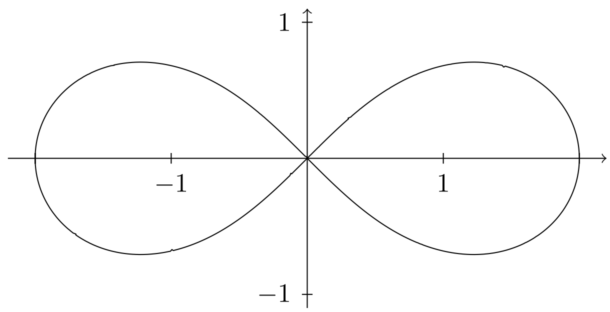
The tangent line to $y = h(x)$ at $x = 1$ is given by

$$y - 16 = 18(x - 1).$$

Problem 3.

Consider the curve described by the equation

$$(x^2 + y^2)^2 = 4(x^2 - y^2).$$



(a) [6pts.] Use the method of implicit differentiation to give a formula for y' in terms of x and y.
[After you differentiate, it might help your algebra to temporarily let $r^2 = (x^2+y^2)$.]

(b) [4pts.] Find the four points where the tangent line to the curve is horizontal.
[None of them have x-coordinate equal to 0; the only point on the curve whose x-coordinate is equal to 0 is $(0, 0)$; your formula for y' is undefined here.]

Solution:

(a) Differentiating both sides gives

$$2(x^2 + y^2)(2x + 2y \cdot y') = 4(2x - 2y \cdot y').$$

Dividing both sides by 4 gives

$$(x^2 + y^2)(x + y \cdot y') = 2x - 2y \cdot y'.$$

Rearranging gives

$$y' = \frac{x(2 - (x^2 + y^2))}{y(2 + (x^2 + y^2))}.$$

(b) We see that $y' = 0$ when $x^2 + y^2 = 2$. We are looking for the points (x, y) with

$$(x^2 + y^2)^2 = 4(x^2 - y^2) \text{ and } x^2 + y^2 = 2.$$

Substituting the second equation into the first gives $4 = 4(x^2 - y^2)$ so that

$$x^2 - y^2 = 1 \text{ and } x^2 + y^2 = 2.$$

We obtain $2x^2 = 3$ and $2y^2 = 1$. The four points are

$$\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{\sqrt{3}}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{\sqrt{3}}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Problem 4.

Calculate the following limits.

(a) [5pts.] $\lim_{x\to 4}\left[(x^2-16)\frac{x-4}{|x-4|}\right]$

(b) [5pts.] $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$

(c) [5pts.] $\lim_{x\to 0}\left[x\sin(\frac{1}{x})\right]$

(d) [5pts.] $\lim_{x\to 3-}\frac{x-4}{x^2-9}$

Solution:

(a) For $x\neq 4$, $\frac{x-4}{|x-4|}$ is equal to either 1 or -1. Thus, for $x\neq 4$,

$$-|x^2-16|\leq\left[(x^2-16)\frac{x-4}{|x-4|}\right]\leq|x^2-16|.$$

Since $\lim_{x\to 4}|x^2-16|=0$ and $\lim_{x\to 4}-|x^2-16|=0$, the squeeze theorem tells us that

$$\lim_{x\to 4}\left[(x^2-16)\frac{x-4}{|x-4|}\right]=0.$$

(b) For $x\neq 4$, we have

$$\begin{aligned}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}&=\frac{x-4}{\sqrt{x}-\sqrt{8-x}}\cdot\frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}}\\&=\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-(8-x)}\\&=\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2(x-4)}=\frac{\sqrt{x}+\sqrt{8-x}}{2}\end{aligned}$$

So $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}=\lim_{x\to 4}\frac{\sqrt{x}+\sqrt{8-x}}{2}=\frac{\sqrt{4}+\sqrt{8-4}}{2}=2.$

(c) Since $-1\leq\sin\left(\frac{1}{x}\right)\leq 1$, we have

$$-|x|\leq x\sin\left(\frac{1}{x}\right)\leq|x|.$$

Since $\lim_{x\to 0}|x|=0$ and $\lim_{x\to 0}(-|x|)=0$, the squeeze theorem tells us that

$$\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right]=0.$$

(d) If x is a number a little less than 3, then $x-4$ is close to -1 and x^2-9 is a small negative number, and so $\frac{x-4}{x^2-9}$ is a big positive number.

$\lim_{x\to 3-}\frac{x-4}{x^2-9}=\infty.$

Problem 5. *2pts.*

This is a bonus question, which is why there are so few points attached to it.

Recall the definition of the derivative of a function $f(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

What is $f'(0)$?

Solution: We have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h})}{h} = \lim_{h\to 0} \left[h \sin\left(\frac{1}{h}\right) \right] = 0.$$