

Math 31A
Differential and Integral Calculus

Final

Instructions: You have 3 hours to complete this exam. There are eight questions, worth a total of ??? points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: _____

Student ID number: _____

Discussion: _____

| Question | Points | Score |
|----------|--------|-------|
| 1 | 0 | |
| 2 | 0 | |
| 3 | 0 | |
| 4 | 0 | |
| 5 | 0 | |
| 6 | 0 | |
| 7 | 0 | |
| 8 | 0 | |
| Total: | 0 | |

Problem 1.

Calculate each of the limits below.

(a) $\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-\sqrt{8-x}}$

Solution:

$$\begin{aligned} \frac{x-4}{\sqrt{x}-\sqrt{8-x}} &= \frac{x-4}{\sqrt{x}-\sqrt{8-x}} \cdot \frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}} \\ &= \frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-(8-x)} = \frac{\sqrt{x}+\sqrt{8-x}}{2} \end{aligned}$$

$$\text{So } \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-\sqrt{8-x}} = \lim_{x \rightarrow 4} \frac{\sqrt{x}+\sqrt{8-x}}{2} = \frac{\sqrt{4}+\sqrt{8-4}}{2} = 2.$$

(b) $\lim_{x \rightarrow \frac{\pi}{2}} (1 - \sin x) \cos(\tan x)$

Solution: We note that $-(1 - \sin x) \leq (1 - \sin x) \cos(\tan x) \leq (1 - \sin x)$.
 $\lim_{x \rightarrow \frac{\pi}{2}} (1 - \sin x) = \lim_{x \rightarrow \frac{\pi}{2}} -(1 - \sin x) = 0$ so the squeeze theorem gives

$$\lim_{x \rightarrow \frac{\pi}{2}} (1 - \sin x) \cos(\tan x) = 0.$$

(c) $\lim_{x \rightarrow 3} \frac{x^2-2}{x^2+x-2}$

Solution: $\frac{3^2-2}{3^2+3-2} = \frac{7}{10}$.

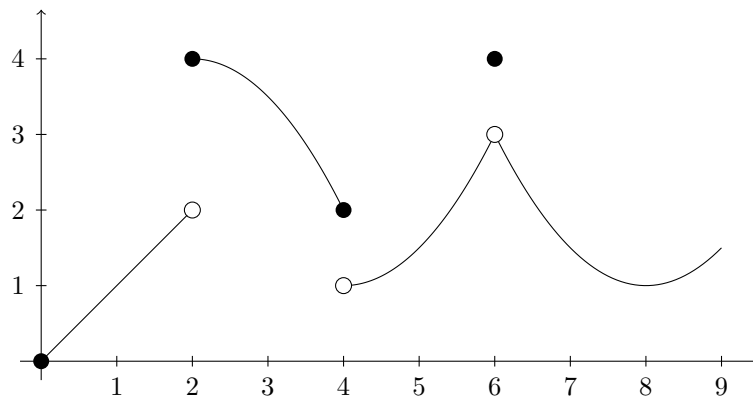
(d) $\lim_{h \rightarrow 0} \frac{\sin(2h)\sin(3h)}{h^2}$

Solution: $\frac{\sin(2h)\sin(3h)}{h^2} = 6 \frac{\sin(2h)}{2h} \frac{\sin(3h)}{3h}$.

So $\lim_{h \rightarrow 0} \frac{\sin(2h)\sin(3h)}{h^2} = 6 \lim_{h \rightarrow 0} \frac{\sin(2h)}{2h} \lim_{h \rightarrow 0} \frac{\sin(3h)}{3h} = 6 \cdot 1 \cdot 1 = 6$.

Problem 2.

Let $f(x)$ be the function shown below.



Answer the following questions. (DNE is a possible answer.)

(a) $f'(1) =$

Solution: $f'(1) = 1.$

(b) $\lim_{x \rightarrow 6} f(x) =$

Solution: $\lim_{x \rightarrow 6} f(x) = 3.$

(c) Is $f(x)$ continuous at $x = 6$?

Solution: No.

(d) Is $f(x)$ left continuous, right continuous or neither at $x = 2$?

Solution: Right continuous.

(e) $\lim_{x \rightarrow 2} f(x)f(x+2)$

Solution: $\lim_{x \rightarrow 2^-} f(x)f(x+2) = 2 \cdot 2 = 4$; $\lim_{x \rightarrow 2^+} f(x)f(x+2) = 4 \cdot 1 = 4$;
so $\lim_{x \rightarrow 2} f(x)f(x+2) = 4.$

Problem 3.

- (a) Find the tangent line to the graph of

$$x^2 + \sin y = xy^2 + 1$$

at $(1, 0)$.

Solution: $2x + (\cos y)y' = y^2 + 2xyy'$ and so

$$2x - y^2 = 2xyy' - (\cos y)y' = (2xy - \cos y)y'.$$

Thus, $y' = \frac{2x - y^2}{2xy - \cos y}$. At $(1, 0)$, $y' = \frac{2}{-1} = -2$.

The tangent line is given by $y = -2(x - 1)$.

- (b) With the same equation as in part a),
are there any points where the tangent line is horizontal?

Solution: The tangent line is horizontal when $y' = 0$, i.e. when $x = \frac{y^2}{2}$.
Substituting this equation into the original one gives

$$\frac{y^4}{4} + \sin y = \frac{y^4}{2} + 1 \text{ so that } \sin y = \frac{y^4}{4} + 1.$$

If $y \neq 0$, then this gives $\sin y > 1$ which is nonsense.

If $y = 0$ we get $0 = 1$, which is nonsense.

There are no points with horizontal tangent line.

Problem 4.

Example 3 of <http://tutorial.math.lamar.edu/Classes/CalcI/RelatedRates.aspx>

Problem 5.

- (a) Let $f(x) = x^3 - 12x$.

Find the global maximum and minimum value of $f(x)$ on the interval $-3 \leq x \leq 5$.

Solution: $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2)$.

Candidate points are $x = -3, -2, 2, \text{ and } 5$.

$$f(-3) = -27 + 36 = 9, \quad f(-2) = -8 + 24 = 16,$$

$$f(2) = 8 - 24 = -16, \quad f(5) = 125 - 60 = 65.$$

$f(x)$ achieves its maximum at $x = 5$, a value of 65.

$f(x)$ achieves its minimum at $x = 2$, a value of -16 .

- (b) Let $f(x) = \frac{x^4}{4} + 2x^3 - \frac{135}{2}x^2$.

I calculated and factored the first derivative: $f'(x) = (x + 15)x(x - 9)$.

Calculate and classify the critical points of $f(x)$.

Find the inflection points of $f(x)$.

It may help your arithmetic to note that $3 \cdot 3 \cdot 3 \cdot 5 = 135$.

Solution: $f'(x) = 0$ when $x = -15, 0$ or 9 . So these are the critical points.

We have

$$f'(x) = x^3 + 6x^2 - 135x.$$

Differentiating and factoring gives

$$f''(x) = 3x^2 + 12x - 135 = 3(x^2 + 4x - 45) = 3(x + 9)(x - 5).$$

$f''(-15) > 0$ so -15 is a local minimum.

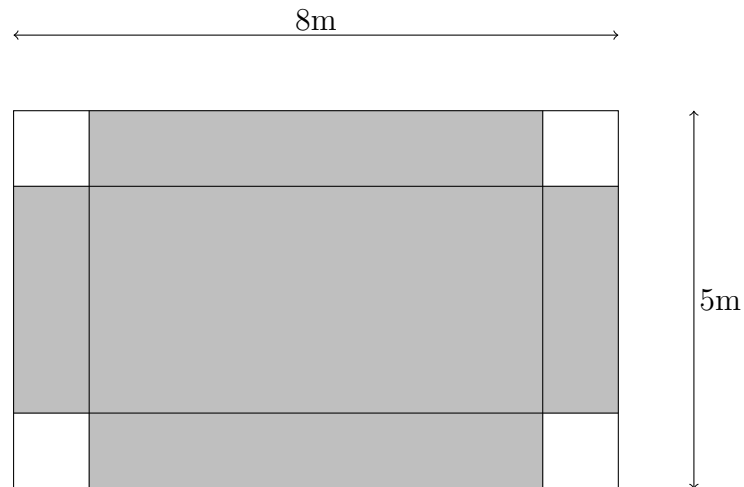
$f''(0) < 0$ so 0 is a local maximum.

$f''(9) > 0$ so 9 is a local minimum.

In addition, this shows we have inflection points at $x = -9$ and $x = 5$.

- (c) Practice questions from the book with asymptotes too!

Problem 6.



You have an 5 inch by 8 inch piece of cardboard.

You make a box by cutting out squares from each corner and folding.

How do you build the box in such a way as to maximize volume?

Help! If you have a quadratic that you are struggling to factor, it is always useful to try some values like -2 , -1 , 0 , 1 , 2 . If plugging in $x = c$ gives zero $(x - c)$ will be a factor.

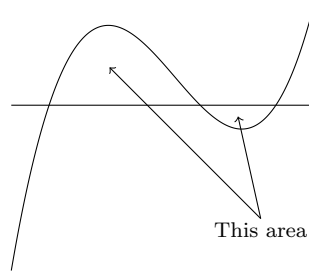
Solution: The volume is given by

$$V(h) = (5 - 2h)(8 - 2h)h = (40 - 26h + 4h^2)h = 40h - 26h^2 + 4h^3.$$

$V'(h) = 40 - 52h + 12h^2 = 4(10 - 13h + 3h^2) = 4(1 - h)(10 - 3h)$. Since $0 \leq h \leq \frac{5}{2}$ we see that $h = 1$ is the only relevant critical point. $V(0) = V(\frac{5}{2}) = 0$ and $V(1) = 18$, so it is a maximum. Use dimensions $6 \times 3 \times 1$.

Problem 7.

Find the area enclosed between the curve $y = 12(x + 2)x(x - 1)$ and the x -axis.

**Solution:**

$$\begin{aligned} & \int_{-2}^0 12(x+2)x(x-1)dx - \int_0^1 12(x+2)x(x-1)dx \\ &= \int_{-2}^0 (12x^3 + 12x^2 - 24x)dx - \int_0^1 (12x^3 + 12x^2 - 24x)dx \\ &= \left[3x^4 + 4x^3 - 12x^2 \right]_{-2}^0 - \left[3x^4 + 4x^3 - 12x^2 \right]_0^1 \\ &= - \left[3(-2)^4 + 4(-2)^3 - 12(-2)^2 \right] - \left[3 + 4 - 12 \right] \\ &= 37. \end{aligned}$$

Problem 8.

- (a) Calculate the indefinite integral $\int \sec x \tan x (\sec x - 1) dx$.

Solution: Let $u = \sec x - 1$. Then $du = (\sec x \tan x) dx$. So

$$\int \sec x \tan x (\sec x - 1) dx = \int u \, du = \frac{u^2}{2} + c = \frac{(\sec x - 1)^2}{2} + c.$$

- (b) Calculate the definite integral

$$\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$$

using the substitution $u = \tan x$ and the trigonometric identity $\tan^2 x + 1 = \sec^2 x$.

Solution: Let $u = \tan x$. Then $u^2 + 1 = \sec^2 x$ and $du = \sec^2 x \, dx$. Thus,

$$\int_0^{\frac{\pi}{4}} \sec^4 x \, dx = \int_0^1 (u^2 + 1) \, du = \left[\frac{u^3}{3} + u \right]_0^1 = \frac{4}{3}.$$