Math 225C: Algebraic Topology

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Contents

1 Fundamental concepts

Definition 1.1. A (locally small) *category* $\mathcal{C}$ consists of the following data:

- a class of *objects* $\mathrm{ob}(\mathcal{C})$;
- for objects $a, b \in \mathcal{C}$, a *hom-set*
$$\mathcal{C}(a,b) = \{f : a \longrightarrow b\}$$
containing all *morphisms* from a to b;
- for objects $a, b, c \in \mathcal{C}$, a *composition* function
$$\mathcal{C}(a,b) \times \mathcal{C}(b,c) \longrightarrow \mathcal{C}(a,c),\ (f,g) \longmapsto g \circ f.$$

We require composition to be associative, and there to exist identity morphisms $1_a \in \mathcal{C}(a,a)$.

Definition 1.2. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories.
A *functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of the following data:

- a (class-)function $F : \mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$, $c \longmapsto Fc$;
- for objects $a, b \in \mathcal{C}$, a function on hom-sets $\mathcal{C}(a,b) \longrightarrow \mathcal{D}(Fa, Fb)$, $f \longmapsto F(f)$.

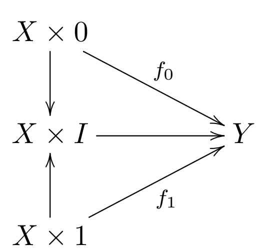
If $f : a \longrightarrow b$ and $g : b \longrightarrow c$, we require that $F(g \circ f) = F(g) \circ F(f)$.
Moreover, for every object $c \in \mathcal{C}$, we demand that $F(1_c) = 1_{Fc}$.

Example 1.3.

1. The free vector space functor $\mathbf{Set} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$.
2. The dual vector space functor $(-)^* : \mathbf{Vect}_{\mathbb{R}}^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$.
3. The forgetful functors $\mathbf{Ab} \longrightarrow \mathbf{Grp} \longrightarrow \mathbf{Set}$.
4. The functor $\mathbf{SmMan} \longrightarrow \mathbf{SmMan}$ which takes M to its tangent bundle TM, and a smooth map f to its derivative Df.
5. The DeRham cochain complex $\Omega^* : \mathbf{SmMan}^{\mathrm{op}} \longrightarrow \mathbf{coCh}_{\mathbb{R}}$.

Definition 1.4. The category of topological spaces $\mathbf{Top}$.
The category of based topological spaces $\mathbf{Top}_*$.

Definition 1.5. $f_0, f_1 : X \longrightarrow Y$ are *homotopic* if there exists a map $F : X \times I \longrightarrow Y$ such that



commutes. F is called a *homotopy*. We often write $F_t : X \longrightarrow Y$ for the restriction of F to $X \times t$.

Suppose $A \subset X$. $f_0, f_1 : X \longrightarrow Y$ are *homotopic rel* A if they are homotopic via a homotopy F such that $f_0|_A = F_t|_A = f_1|_A$ for all $t \in I$.

$f_0, f_1 : (X, x) \to (Y, y)$ are *based homotopic* if the underlying maps $f_0, f_1 : X \to Y$ are homotopic rel x.

Definition 1.6. The homotopy category of space $\mathbf{Ho}(\mathbf{Top})$.

The homotopy category of based space $\mathbf{Ho}(\mathbf{Top}_*)$.

The isomorphisms in $\mathbf{Ho}(\mathbf{Top})$ are called *homotopy equivalences*.

The isomorphisms in $\mathbf{Ho}(\mathbf{Top})$ are called *based homotopy equivalences*.

Definition 1.7. A space X is called a *CW-complex* if it can be written as a union $\bigcup_{n \in \mathbb{N}} X^n$ and the following conditions hold.

1. X^0 is a discrete topological space.

2. The n-skeleton X^n is built from X^{n-1} by attaching n-cells. That is,

    - we have a collection of *attaching maps* $\{\varphi_\alpha : S^{n-1}_\alpha \longrightarrow X^{n-1}\}$;
    - $X^n = \frac{\bigsqcup_\alpha D^n_\alpha \sqcup X^{n-1}}{x \in \partial D^n_\alpha \sim \varphi_\alpha(x)}$.

3. $A \subset X$ is closed if and only if for all $n \in \mathbb{N}$, $A \cap X_n$ is closed in X_n.

A based CW-complex $(X, *)$ is a CW-complex together with a choice of basepoint $* \in X^0$.

Definition 1.8. Suppose X is a CW-complex.

X^n is called the *n-skeleton* of X. By convention, $X^{-1} = \emptyset$ (or $*$ in the based setting).

As a set, $X^n = \bigsqcup_\alpha e^n_\alpha \sqcup X^{n-1}$, where $e^n_\alpha = D^n_\alpha \setminus \partial D^n_\alpha$. e^n_α is called an *n-cell*.

We have *characteristic maps* $\Phi : D^n_\alpha \longrightarrow X^n$ making the following diagram commute:

$$\begin{array}{ccc} \bigsqcup_\alpha S^{n-1}_\alpha & \xrightarrow{\bigsqcup_\alpha \varphi_\alpha} & X^{n-1} \\ \downarrow & & \downarrow \\ \bigsqcup_\alpha D^n_\alpha & \xrightarrow{\bigsqcup_\alpha \Phi_\alpha} & X^n. \end{array}$$

In fact, this diagram is a pushout in $\mathbf{Top}$ (in case you know more category theory). Moreover, X is the colimit of the X^ns.

2 The functor $\pi_1 : \mathbf{Ho}(\mathbf{Top}_*) \longrightarrow \mathbf{Grp}$

Definition 2.1. Suppose X is a space. A *path* in X is a map $p : I \longrightarrow X$.

Two paths $p_0, p_1 : I \longrightarrow X$ are *path homotopic* if they are homotopic rel ∂I. We write $p_0 \simeq p_1$.

A path $l : I \longrightarrow X$ with $l(0) = l(1)$ is called a *loop*.

Definition 2.2. Suppose $p, q : I \longrightarrow X$ are paths and that $p(1) = q(0)$. We write $p \cdot q : I \longrightarrow X$ for the path

$$(p \cdot q)(x) = \begin{cases} p(2s) & \text{if } s \in [0, \frac{1}{2}]; \\ q(2s-1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

$p \cdot q$ is called the *concatenation* of p and q. We write $\overline{p} : I \longrightarrow X$ for the *inverse path* $\overline{p}(s) = p(1-s)$.

Lemma 2.3. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

Definition 2.4. Suppose (X, x) is a based space. Then $\pi_1(X, x)$ is the set of equivalence classes of loops starting (and ending) at x.

Theorem 2.5. *Suppose (X, x) is a based space. Then $\pi_1(X, x)$ is a group under the operation*

$$[l][l'] = [l \cdot l'].$$

The constant loop $c : I \longrightarrow x \longrightarrow X$ is the identity element.

We call $\pi_1(X, x)$ the fundamental group.

Theorem 2.6. *Suppose $f : (X, x) \longrightarrow (Y, y)$ is a map of based spaces. Then*

$$f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y), \ [l] \longrightarrow [f \circ l]$$

is a group homomorphism. We call f_* the homomorphism induced by f.

Moreover, π_1 defines a functor

$$\textbf{\textit{Ho}}(\textbf{\textit{Top}}_*) \longrightarrow \textbf{\textit{Grp}}.$$

It is a categorical argument to see that a based homotopy equivalence induces an isomorphism on fundamental groups. Unbased homotopy equivalences also induce isomorphisms on fundamental groups but seeing this requires a little more thought.

Proposition 2.7. *Suppose X is a space, $x_0, x_1 \in X$, and that $p : I \longrightarrow X$ is a path from x_0 to x_1.*

Then

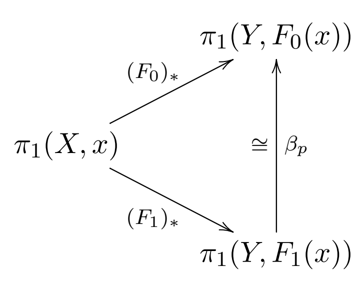
$$\beta_p : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0), \ [l] \longmapsto [p \cdot l \cdot \overline{p}]$$

is an isomorphism of groups.

Definition 2.8. We call a space X *simply-connected* if it is path-connected and has trivial fundamental group.

Proposition 2.9. *A space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

Lemma 2.10. *Suppose X, Y are spaces, $x \in X$, and $F : X \times I \to Y$ is a homotopy. Let $p : I \longrightarrow X$ be defined by $p(t) = F(x, t)$. Then*



commutes. In particular, $(F_0)_*$ is an isomorphism if and only if $(F_1)_*$ is an isomorphism.

Theorem 2.11. *Suppose X and Y are spaces, $x \in X$, and $f : X \longrightarrow Y$ is a homotopy equivalence. Then the induced homomorphism $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ is an isomorphism.*

3 Basic covering space theory and $\pi_1(S^1, *)$

Definition 3.1. Suppose X is a space. A *covering space* of X is a space $\tilde{X}$ together with a map $\pi : \tilde{X} \longrightarrow X$. The map is required to have the following property:

- for all $x \in X$, there exists an open set U_x in X containing x such that

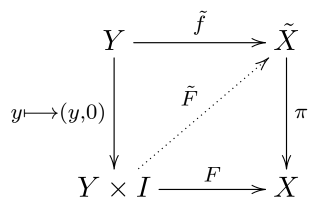
$$\pi^{-1}(U_x) = \bigsqcup_{\tilde{x} \in \pi^{-1}(x)} U_{\tilde{x}}$$

  and for all $\tilde{x} \in \pi^{-1}(x)$, $\pi|_{U_{\tilde{x}}} : U_{\tilde{x}} \longrightarrow U_x$ is a homeomorphism.

Such a U_x is called an *evenly covered neighborhood of x*.

Example 3.2. $\pi : \mathbb{R} \longrightarrow S^1$, $t \longmapsto (\cos(2\pi t), \sin(2\pi t))$ is a covering map.

Theorem 3.3. *Suppose $\pi : \tilde{X} \longrightarrow X$ is a covering space.*



Given a homotopy $F : Y \times I \to X$, and a map $\tilde{f} : Y \to \tilde{X}$ lifting F_0, there exists a unique homotopy $\tilde{F} : Y \times I \longrightarrow \tilde{X}$ lifting F with $\tilde{F}_0 = \tilde{f}$.

Proof. First, fix $y_0 \in Y$. We'll construct the lift near y_0.

Let $t \in I$. We can find an open $V_t \subset Y$ containing y_0 and an open (in I) interval W_t containing t such that $F(V_t \times W_t)$ is contained in an evenly covered neighborhood of $F(y_0, t)$. Compactness implies that we can cover $y_0 \times I$ with finitely sets from $\{V_t \times W_t : t \in I\}$. Taking the relevant finite intersection of the V_t's we obtain an open $V \subset Y$ containing y_0. Using the finitely many W_t's, we obtain a partition of I: $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$. For each $i \in \{1, \ldots, n\}$, $F(V \times [t_{i-1}, t_i])$ is contained in an evenly covered neighborhood U_i.

Assume we've defined $\tilde{F}$ on $V \times [0, t_{i-1}]$. We have $F(V \times [t_{i-1}, t_i]) \subset U_i$, and a $\tilde{U}_i$ containing $\tilde{F}(y_0, t_{i-1})$ mapping homeomorphically to U_i. We would like to define $\tilde{F}$ on $V \times [t_{i-1}, t_i]$ to be F composed with the inverse homeomorphism. To make sure this agrees with what has already been defined we should replace V by $\tilde{F}|^{-1}_{V \times t_{i-1}}(\tilde{U}_i)$. After finitely many steps, this defines $\tilde{F}$ on $V \times I$ for some open set V containing y_0.

First, argue uniqueness on $y_0 \times I$. This means that by varying y_0, the construction above defines $\tilde{F}$ uniquely. □

Corollary 3.4. *Suppose $\pi : \tilde{X} \longrightarrow X$ is a covering space. Let $p : I \longrightarrow X$ be a path and $x_0 = p(0)$. For every lift $\tilde{x}_0$ of x_0, there is a unique path $\tilde{p} : I \longrightarrow \tilde{X}$ lifting p with $\tilde{p}(0) = \tilde{x}_0$.*

Corollary 3.5. *Suppose $\pi : \tilde{X} \longrightarrow X$ is a covering space. Given a homotopy of paths $p_t : I \longrightarrow X$, and a lift $\tilde{p}_0 : I \longrightarrow \tilde{X}$ of the path p_0, there exists a unique homotopy $\tilde{p}_t : I \longrightarrow \tilde{X}$ of paths lifting p_t which starts at $\tilde{p}_0$.*

Theorem 3.6. *Let $* = (1,0) \in S^1$. We have an isomorphism $\mathcal{I}_{S^1} : \mathbb{Z} \longrightarrow \pi_1(S^1, *)$ defined by*

$$n \longmapsto \Big[t \longmapsto (\cos(2\pi nt), \sin(2\pi nt))\Big].$$

Proof. When $n \in \mathbb{Z}$, write ω_n for the loop $t \longmapsto (\cos(2\pi nt), \sin(2\pi nt))$, so $\mathcal{I}_{S^1}(n) = [\omega_n]$.

One checks $\mathcal{I}_{S^1}$ is a homomorphism.

We make use of the covering map $\pi : (\mathbb{R}, 0) \longrightarrow (S^1, *)$.

First, we show that $\mathcal{I}_{S^1}$ is surjective. Suppose $l : (I, \partial I) \longrightarrow (S^1, *)$ is a loop. l lifts uniquely to a path $p : (I, 0) \longrightarrow (\mathbb{R}, 0)$. Note that $p(1) \in \pi^{-1}(*) = \mathbb{Z}$; let $n = p(1)$. We claim $\mathcal{I}_{S^1}(n) = [l]$. To see this, note that p is homotopic via the straight line homotopy to the path $t \longmapsto nt$. Composing this homotopy with π, we obtain a homotopy from l to ω_n.

We now show that $\mathcal{I}_{S^1}$ is injective. Suppose $[\omega_n] = [\omega_m]$. Then $\omega_n \simeq \omega_m$, and we can lift such a homotopy to a homotopy of paths in $\mathbb{R}$ starting at 0. The endpoints must be fixed throughout the homotopy, and this shows $n = m$. □

4 The fundamental group of a CW-complex

Since $\{0\} \longrightarrow \mathbb{R}^n$ and $\mathbb{R}^n \longrightarrow \{0\}$ are homotopy inverses we have:

Theorem 4.1. *$\mathbb{R}^n$ is simply-connected.*

Theorem 4.2. *For $n \geq 2$, S^n is simply-connected.*

Proof. Recall $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Let

$$E = \{(x_1, \ldots, x_n, x_{n+1}) \in S^n : x_{n+1} = 0\},\ H_U = \{(x_1, \ldots, x_n, x_{n+1}) \in S^n : x_{n+1} > 0\},$$

$p_N = (0, \ldots, 0, 1)$, and $p_S = (0, \ldots, 0, -1)$. Stereographic projection defines a homeomorphism $(S^n \setminus p_N, p_S) \longrightarrow (\mathbb{R}^n, 0)$. Thus, $\pi_1(S^n \setminus p_N, p_S) = 0$. To prove the theorem, it is enough to show that the inclusion $S^n \setminus p_N \longrightarrow S^n$ induces a surjection $\pi_1(S^n \setminus p_N, p_S) \longrightarrow \pi_1(S^n, p_S)$.

Let $l : I \longrightarrow S^n$ be a loop and note $l^{-1}(p_N) \subset l^{-1}(H_U) \subset I$. $l^{-1}(H_U)$ is open, so can be written as

$$l^{-1}(H_U) = \bigsqcup_{j \in J} (a_j, b_j).$$

$l^{-1}(p_N)$ is compact and so $J' := \{j \in J :\ (a_j, b_j) \cap l^{-1}(p_N) \neq \emptyset\}$ is finite. For each $j \in J'$, consider $l|_{[a_j, b_j]}$. Its image is contained in $E \cup H$, and $l(a_j), l(b_j) \in E$. E is homeomorphic to S^{n-1} which is path-connected because $n \geq 2$. Moreover, $E \cup H$ is homeomorphic to D^n which is simply-connected. These two facts mean that we can homotope $l|_{[a_j, b_j]}$ rel $\partial[a_j, b_j]$ to a path in E. Doing this for all $j \in J'$, defines a homotopy from l to another loop whose image is in $S^n \setminus p_N$. □

The previous theorem is also a consequence of Van Kampen's theorem which we'll see shortly.

Theorem 4.3. *Suppose $(X, *)$ is a space, that $n \geq 2$, and that we have a map $\varphi : S^{n-1} \longrightarrow X$. Let Y be the space obtained by attaching an n-cell to X using φ. Then the inclusion $X \longrightarrow Y$ induces a surjection $\pi_1(X, *) \longrightarrow \pi_1(Y, *)$.*

Proof. Let $\Phi : D^n \longrightarrow Y$ be the characteristic map of the n-cell, let $y = \Phi(0)$, and notice that $Y \setminus y$ deformation retracts onto X.

$$\begin{array}{ccc} \pi_1(X,*) & & \\ \Big\downarrow \cong & \searrow & \\ & & \pi_1(Y,*) \\ & \nearrow & \\ \pi_1(Y \setminus y, *) & & \end{array}$$

Thus, the inclusion $X \longrightarrow Y \setminus y$ is a homotopy equivalence, and induces an isomorphism $\pi_1(X,*) \longrightarrow \pi_1(Y \setminus y, *)$. We just have to show $\pi_1(Y \setminus y, *) \longrightarrow \pi_1(Y,*)$ is surjective. This is the same argument as the previous theorem: let $l : I \longrightarrow Y$ be a loop and note $l^{-1}(y) \subset l^{-1}(\frac{1}{2}e^n) \subset I$. □

Theorem 4.4. *Suppose $(X,*)$ is a space, and that $\{\varphi_\alpha : S_\alpha^{n_\alpha - 1} \longrightarrow X\}$ is a collection of attaching maps with each $n_\alpha \geq 2$. Let Y be the space obtained by attaching cells to X using $\{\varphi_\alpha\}$. Then the inclusion $X \longrightarrow Y$ induces a surjection $\pi_1(X,*) \longrightarrow \pi_1(Y,*)$.*

Proof. Same as before, but deal with all cells simultaneously. □

Theorem 4.5. *Suppose $(X,*)$ is a CW-complex. Then the inclusions $X^1 \longrightarrow X^2$ and $X^2 \longrightarrow X$ induce surjections $\pi_1(X^1,*) \longrightarrow \pi_1(X^2,*)$ and $\pi_1(X^2,*) \longrightarrow \pi_1(X,*)$, respectively.*

Proof. The previous theorem tells us that $\pi_1(X^1,*) \longrightarrow \pi_1(X^2,*)$ is surjective.

Suppose $l : (I, \partial I) \to (X,*)$ is a loop. Since its image is compact, it lies entirely within some n-skeleton X^n. Thus, its homotopy class is in the image of $\pi_1(X^n,*) \longrightarrow \pi_1(X,*)$. The previous theorem tells us that each of the maps

$$\pi_1(X^2,*) \longrightarrow \pi_1(X^3,*) \longrightarrow \ldots \longrightarrow \pi_1(X^n,*)$$

is surjective, so $[l]$ is in the image of $\pi_1(X^2,*) \longrightarrow \pi_1(X,*)$. □

In order to be able to calculate fundamental groups, we need to prove the following theorems. Theorems 4.6, 4.10, 4.12 can be proved using Van Kampen's theorem which we will mention shortly; they can also be proved using covering space theory.

Theorem 4.6. *Suppose $(X,*)$ is a space, that $n \geq 3$, and that we have a map $\varphi : S^{n-1} \longrightarrow X$. Let Y be the space obtained by attaching an n-cell to X using φ. Then the inclusion $X \longrightarrow Y$ induces an injection $\pi_1(X,*) \longrightarrow \pi_1(Y,*)$. In fact, in light of theorem 4.3, it induces an isomorphism.*

Theorem 4.7. *Suppose $(X,*)$ is a space, and that $\{\varphi_\alpha : S_\alpha^{n_\alpha - 1} \longrightarrow X\}$ is a collection of attaching maps with each $n_\alpha \geq 3$. Let Y be the space obtained by attaching cells to X using $\{\varphi_\alpha\}$. Then the inclusion $X \longrightarrow Y$ induces an injection $\pi_1(X,*) \longrightarrow \pi_1(Y,*)$.*

In fact, in light of theorem 4.4, it induces an isomorphism.

Proof. Same as the proof just omitted before, but deal with all cells simultaneously. □

Theorem 4.8. *Suppose $(X,*)$ is a CW-complex. Then the inclusion $X^2 \longrightarrow X$ of the 2-skeleton induces an injection $\pi_1(X^2,*) \longrightarrow \pi_1(X,*)$.*

In fact, in light of theorem 4.5, it induces an isomorphism.

Proof. Suppose $l : (I, \partial I) \to (X^2, *)$ is a loop and that its image under $\pi_1(X^2, *) \longrightarrow \pi_1(X, *)$ is 0, that is, it is nullhomotopic in X. Since the image of the nullhomotopy is compact, it lies entirely within some n-skeleton X^n. Thus, the image of $[l]$ under $\pi_1(X^2, *) \longrightarrow \pi_1(X^n, *)$ is 0. The previous theorem tells us that each of the maps

$$\pi_1(X^2, *) \longrightarrow \pi_1(X^3, *) \longrightarrow \ldots \longrightarrow \pi_1(X^n, *)$$

is injective, so $[l]$ is 0 in $\pi_1(X^2, *)$. □

Remark 4.9. The map $I \to S^1$, $t \longmapsto (\cos(2\pi t), \sin(2\pi t))$ induces a homeomorphism $I/\partial I \to S^1$.
Thus, a loop $l : (I, \partial I) \longrightarrow (X, x_0)$ is the same as a map $\varphi : (S^1, *) \longrightarrow (X, x_0)$.

Theorem 4.10. *Suppose (X, x_0) is a based space, that $p : I \longrightarrow X$ is a path with $p(0) = x_0$, and that $\varphi : (S^1, *) \longrightarrow (X, p(1))$. Let Y be the space obtained by attaching a 2-cell to X using φ, and l be the loop corresponding to φ. Then the kernel of $\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$ is the normal subgroup generated by $[p \cdot l \cdot \overline{p}]$.*

Theorem 4.11. *Suppose (X, x_0) is a based space, that $\{p_\alpha : I \longrightarrow X\}$ is a collection of paths with $p_\alpha(0) = x_0$ for all α, and that $\{\varphi_\alpha : (S^1_\alpha, *) \longrightarrow (X, p_\alpha(1))\}$ is a collection of attaching maps. Let Y be the space obtained by attaching 2-cells to X using $\{\varphi_\alpha\}$, and $\{l_\alpha\}$ be the loops corresponding to $\{\varphi_\alpha\}$. Then the kernel of $\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$ is the normal subgroup generated by $\{[p_\alpha \cdot l_\alpha \cdot \overline{p}_\alpha]\}$.*

Proof. Same as the proof just omitted before, but deal with all cells simultaneously. □

Recall theorem 3.6. We also have the following result.

Theorem 4.12. *We have an isomorphism*

$$*_\alpha \, \mathbb{Z} \longrightarrow \pi_1\Big(\bigvee_\alpha S^1, *\Big).$$

The map is determined by the fact that the following diagram commutes for all α:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\mathcal{I}_{S^1}} & \pi_1(S^1, *) \\ {\scriptstyle i_\alpha}\downarrow & & \downarrow{\scriptstyle (i_\alpha)_*} \\ *_\alpha\mathbb{Z} & \longrightarrow & \pi_1\big(\bigvee_\alpha S^1, *\big). \end{array}$$

Theorem 4.13. *The 1-skeleton of a path-connected CW-complex is homotopy equivalent to a wedge of circles.*

Proof. A tree is contractible. Quotienting out by a maximal subtree is a homotopy equivalence to a wedge of circles. □

Corollary 4.14. *Suppose $(X, *)$ is a path-connected CW-complex. Then $\pi_1(X, *)$ has a presentation in which the relations correspond to the attaching maps of 2-cells.*

5 Van Kampen's Theorem

Definition 5.1. *Free product of groups* as in Hatcher page 41.

Suppose $(X, *)$ is a based space and that A, B are subsets of X containing $*$. Then we have a commutative diagram induced by inclusions:

$$\begin{CD} \pi_1(A \cap B, *) @>{(i_{BA})_*}>> \pi_1(B, *) \\ @V{(i_{AB})_*}VV @VV{(j_B)_*}V \\ \pi_1(A, *) @>{(j_A)_*}>> \pi_1(X, *). \end{CD} \tag{5.2}$$

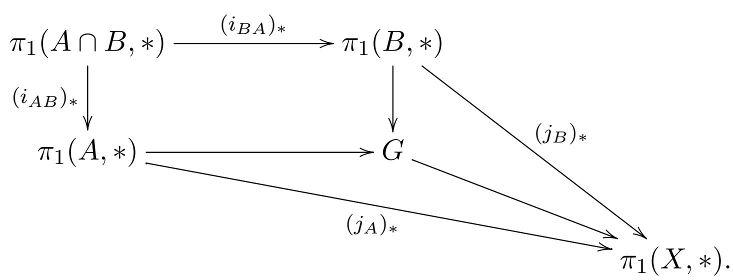
Suppose we started with just

$$\begin{CD} \pi_1(A \cap B, *) @>{(i_{BA})_*}>> \pi_1(B, *) \\ @V{(i_{AB})_*}VV \\ \pi_1(A, *) \end{CD}$$

and wanted to create a commutative square. We could take

$$G = \frac{\pi_1(A, *) * \pi_1(B, *)}{\left\langle (i_{AB})_*(\omega)^{-1}(i_{BA})_*(\omega) : \ \omega \in \pi_1(A \cap B, *) \right\rangle}.$$

In fact, this group is the "best" such group in the sense that commutativity of (5.2) guarantees the existence of a unique homomorphism $G \longrightarrow \pi_1(X, *)$ such that the composites $\pi_1(A, *) \longrightarrow G \longrightarrow \pi_1(X, *)$ and $\pi_1(B, *) \longrightarrow G \longrightarrow \pi_1(X, *)$ are $(j_A)_*$ and $(j_B)_*$, respectively:



Theorem 5.3 (Van Kampen)**.** *Suppose that $(X, *)$ is a based space and that A, B are subsets of X containing $*$. Suppose, in addition, that A, B, and $A \cap B$ are open and path-connected, and that $A \cup B = X$. Then the map*

$$\frac{\pi_1(A, *) * \pi_1(B, *)}{\left\langle (i_{AB})_*(\omega)^{-1}(i_{BA})_*(\omega) : \ \omega \in \pi_1(A \cap B, *) \right\rangle} \longrightarrow \pi_1(X)$$

determined by $(j_A)_*$ and $(j_B)_*$ is an isomorphism.

Proof. Surjectivity isn't so bad.

Let $l : (I, \partial I) \longrightarrow (X, *)$ be a loop. Using compactness of I, we can find a partition

$$0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = 1$$

such that for all $i \in \{1, \ldots, n\}$, $f([s_{i-1}, s_i])$ is a subset of either A or B. Since $A \cap B$ is path-connected, we can find paths $p_i : I \longrightarrow A \cap B$, with $p_i(0) = *$ and $p_i(1) = f(s_i)$. We have

$$[l] = \left[p_0 \cdot f|_{[s_0,s_1]} \cdot \overline{p_1}\right] \left[p_1 \cdot f|_{[s_1,s_2]} \cdot \overline{p_2}\right] \cdots \left[p_{n-1} \cdot f|_{[s_{n-1},s_n]} \cdot \overline{p_n}\right]$$

which is in the image of the map determined by $(j_A)_*$ and $(j_B)_*$.

Injectivity is more difficult. See Hatcher for a proof. □

Example 5.4. $\pi_1(S^1 \vee S^1, *) \cong \mathbb{Z} * \mathbb{Z}$.

Theorem 4.12 follows by induction for finite indexing sets. It follows for arbitrary indexing sets from the stronger version of Van Kampen's theorem stated in Hatcher.

We turn to the proofs of theorem 4.10 and theorem 4.6.

Proof of theorem 4.10 and theorem 4.6. Suppose (X, x_0) is a based space. Because the fundamental group of (X, x_0) only "sees" the path-component of x_0, we may as well assume that X is path-connected.

Suppose $p : I \to X$ is a path with $p(0) = x_0$, and $\varphi : (S^1, *) \longrightarrow (X, p(1))$. Let Y be the space obtained by attaching a 2-cell to X using φ, $\Phi : D^2 \longrightarrow Y$ be the characteristic map of the 2-cell, and $y = \Phi(0)$. Let l be the loop corresponding to φ. To see that the kernel of $\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$ is the normal subgroup generated by $[p \cdot l \cdot \overline{p}]$, we will apply Van Kampen's theorem to a space which deformation retracts to Y. The reason for fattening up Y is that we do not know what Y looks like near x_0, and so describing opens sets containing x_0 is not possible.

Let

$$Z = \frac{(I \times I) \sqcup Y}{(s, 0) \in I \times I \sim p(s) \in Y,\ (1, t) \in I \times I \sim \Phi(1 - \frac{t}{2}, 0) \in Y},$$

$z_0 \in Z$ be the image of $(0, 1) \in I \times I$, X' be the image of $(0 \times I) \sqcup X$, Y' be the image of $(0 \times I) \sqcup Y$, $A = Z - y$, and $B = Z \setminus X$. Let C be the image of $I \times 1 \sqcup \Phi(\frac{1}{2}S^1)$ and c_0 be the image of $(1, 1)$.

The inclusions $X \to X'$, $Y \to Y'$, $X' \to A$, $z_0 \to B$, $\Phi(\frac{1}{2}-) : S^1 \to C$, $C \to A \cap B$, and $Y' \to Z$ are homotopy equivalences, we have an obvious path q_1 from z_0 to x_0 in X', and an obvious path q_2 from z_0 to c_0 in C. Thus, we obtain the following commutative diagram.

$$\begin{array}{ccccc}
\pi_1(S^1, *) \cong \mathbb{Z} & \xrightarrow{[p \cdot l \cdot \overline{p}]} & \pi_1(X, x_0) & \longrightarrow & \pi_1(Y, x_0) \\
\Phi(\frac{1}{2}-)_* \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\pi_1(C, c_0) & & \pi_1(X', x_0) & \longrightarrow & \pi_1(Y', x_0) \\
\beta_{q_2} \downarrow \cong & & \beta_{q_1} \downarrow \cong & & \beta_{q_1} \downarrow \cong \\
\pi_1(C, z_0) & & \pi_1(X', z_0) & \longrightarrow & \pi_1(Y', z_0) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\pi_1(A \cap B, z_0) & \longrightarrow & \pi_1(A, z_0) & \longrightarrow & \pi_1(Z, z_0)
\end{array}$$

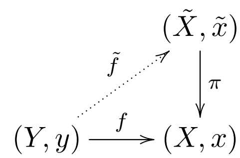
Since $\pi_1(B, z_0) = 0$, Van Kampen's theorem says $\pi_1(Z, z_0)$ is the quotient of $\pi_1(A, z_0)$ by the normal subgroup generated by the image of $\pi_1(A \cap B, z_0)$. This fact together with the commutative diagram finishes the proof of 4.10. The same proof works for 4.6, but in that case $\pi_1(A \cap B, z_0) = 0$. $\square$

6 Further covering space theory

Proposition 6.1. *Suppose $\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ is a covering map.*

The induced map $\pi_* : \pi_1(\tilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x)$ is injective, and its image consists of the homotopy classes of loops in X based at x whose lifts to $\tilde{X}$ starting at $\tilde{x}$ are loops.

Proposition 6.2. *Suppose $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering space and that $f : (Y, y) \to (X, x)$ is a map where Y is path-connected and locally path-connected.*



Then a lift $\tilde{f} : (Y, y) \longrightarrow (\tilde{X}, \tilde{x})$ exists if and only if $f_*(\pi_1(Y, y)) \subset \pi_*(\pi_1(\tilde{X}, \tilde{x}))$.

Proof. If a lift $\tilde{f}$ exists then $f_*(\pi_1(Y, y)) = \pi_* \tilde{f}_*(\pi_1(Y, y)) \subset \pi_*(\pi_1(\tilde{X}, \tilde{x}))$.

Conversely, suppose $f_*(\pi_1(Y, y)) \subset \pi_*(\pi_1(\tilde{X}, \tilde{x}))$. Given $y' \in Y$, we can find a path $p : I \longrightarrow Y$ from y to y'. Lift fp starting at $\tilde{x}$ to obtain $\widetilde{fp} : I \longrightarrow \tilde{X}$. Define $\tilde{f}(y') = \widetilde{fp}(1)$.

First, we argue that $\tilde{f}$ is well-defined. Suppose $q : I \longrightarrow Y$ is a second path from y to y'. Then $p \cdot \overline{q}$ is a loop based at y. So $fp \cdot \overline{fq}$ is loop based at x in $f_*(\pi_1(Y, y))$, thus, in $\pi_*(\pi_1(\tilde{X}, \tilde{x}))$. Therefore $fp \cdot \overline{fq}$ lifts to a loop in $\tilde{X}$ based at $\tilde{x}$. We conclude that $\widetilde{fp}(1) = \widetilde{fq}(1)$. So $\tilde{f}$ is well-defined.

The map is continuous because locally it looks like $\pi^{-1} f$. $\square$

Proposition 6.3. *Suppose $\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ is a covering space and $f : (Y, y) \longrightarrow (X, x)$ is a map from a connected space Y. Any two lifts $\tilde{f}_1, \tilde{f}_2 : (Y, y) \longrightarrow (\tilde{X}, \tilde{x})$ of f are equal.*

Proposition 6.4. *Suppose (X, x) is path-connected and locally path-connected. Two path-connected covering spaces $p_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x)$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x)$ are isomorphic if and only if*

$$(p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2)).$$

Theorem 6.5. *Suppose (X, x) is path-connected, locally path-connected, and semilocally simply-connected. Then there exists a covering map $\pi_0 : (\tilde{X}_0, \tilde{x}_0) \to (X, x)$ in which $\tilde{X}_0$ is simply-connected.*

Proof. Suppose X is path-connected, locally path-connected, and semilocally simply-connected.

Let

$$\tilde{X}_0 = \{[p] : p \text{ is a path starting at } x\},$$

where $[p]$ denotes the homotopy class of p with respect to homotopies that fix the end-points. Let $\tilde{x}_0$ the homotopy class of the constant path starting at x. Let $\pi_0([p]) := p(1)$.

It is possible to put a topology on $\tilde{X}_0$ so that $\pi_0 : (\tilde{X}_0, \tilde{x}_0) \longrightarrow (X, x)$ is a covering map.

Given $[p] \in \tilde{X}_0$. Let $p_t : I \to X$ be the path such that $p_t|_{[0,t]} = p|_{[0,t]}$ and p_t is constant on $[t, 1]$. Then $t \longmapsto [p_t]$ is a path from $\tilde{x}_0$ to $[p]$ in $\tilde{X}_0$. Thus, $\tilde{X}_0$ is path-connected.

Let $l : (I, \partial I) \longrightarrow (X, x)$ be a loop. $\tilde{p} : t \longmapsto [l_t]$ is a lift of l to $\tilde{X}_0$ starting at $\tilde{x}_0$. If $[l]$ is in the image of $(\pi_0)_*$, then $\tilde{p}$ is a loop, so $[l] = [l_1] = \tilde{p}(1) = \tilde{x}_0$, which means l is nullhomotopic. Thus, the image of $\pi_1(\tilde{X}_0, \tilde{x}_0)$ under $(\pi_0)_*$ is 0. So $\pi_1(\tilde{X}_0, \tilde{x}_0) = 0$, and $\tilde{X}_0$ is simply-connected. □

Definition 6.6. Suppose (X, x) is path-connected, locally path-connected, and semilocally simply-connected. The $(\tilde{X}_0, \tilde{x}_0)$ constructed in the previous theorem is called the *universal cover* of (X, x).

The construction gives $(\tilde{X}_0, \tilde{x}_0)$ a left $\pi_1(X, x)$-action: $[l][p] = [l \cdot p]$.

Remark 6.7. From now on we'll always assume (X, x) is path-connected, locally path-connected, and semilocally simply-connected, so that it has a universal cover.

Definition 6.8. Suppose $\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ is a covering map.

Suppose $f \in \pi^{-1}(x)$ and that $[l] \in \pi_1(X, x)$. By lifting l to a path starting at f and recording where the path ends, we obtain another point $f' \in \pi^{-1}(x)$. Define

$$f[l] := f'.$$

Thus, $\pi^{-1}(x)$ is a right $\pi_1(X, x)$-set with $\tilde{x}$ as a distinguished point.

This is a called *the action of* $\pi_1(X, x)$ *on the fiber* $\pi^{-1}(x)$.

This procedure defines a functor **Fib** from $\mathbf{Cov}_*(X, x)$ to $\pi_1(X, x)$-$\mathbf{Set}_*$.

Definition 6.9. Suppose (X, x) is a based space and that F is a $\pi_1(X, x)$-set with a distinguished point $*$. Then we can construct a covering map $\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ as follows.

Let $\pi_0 : (\tilde{X}_0, \tilde{x}_0) \longrightarrow (X, x)$ be the universal cover of (X, x). Let

$$\tilde{X} = \frac{F \times \tilde{X}_0}{(f \cdot [l], [p]) \sim (f, [l \cdot p])}$$

and $\tilde{x} = [(*, \tilde{x}_0)]$. Define $\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ by $[(f, [p])] \longmapsto p(1)$.

This is called the *Borel construction.*

Notice that $\pi^{-1}(x) = F \times \tilde{x}_0 / \sim$.

This procedure defines a functor **Bor** from $\pi_1(X, x)$-$\mathbf{Set}_*$ to $\mathbf{Cov}_*(X, x)$.

Theorem 6.10. ***Fib*** *and* ***Bor*** *define an equivalence of categories* $\mathbf{Cov}_*(X, x) \simeq \pi_1(X, x)$-$\mathbf{Set}_*$.

Proof. Suppose $\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ is a covering map.

Given $p : I \longrightarrow X$ starting at x and $f \in \pi^{-1}(x)$, we can lift p starting at f, to obtain $\tilde{p} : I \longrightarrow \tilde{X}$. Then $\tilde{p}(1) \in \tilde{X}$. We can define

$$\frac{\pi^{-1}(x) \times \tilde{X}_0}{(f \cdot [l], [p]) \sim (f, [l \cdot p])} \longrightarrow \tilde{X}, \;\; [(f, [p])] \longmapsto \tilde{p}(1).$$

This is a homeomorphism (exercise).

Conversely, suppose (X, x) is a based space and that F is a $\pi_1(X, x)$-set and construct

$$\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$$

as in definition 6.9.

We have already remarked that $F \to \pi^{-1}(x)$, $f \longmapsto [(f, \tilde{x}_0)]$ is a bijection of sets. We show that it is an isomorphism of $\pi_1(X, x)$-sets. Suppose $f \in F$ and that $[l] \in \pi_1(X, x_0)$. Then $p : I \longrightarrow \tilde{X}$, $t \longmapsto [(f, [l_t])]$ is a path from $[(f, \tilde{x}_0)]$ to $[(f, [l])] = [(f[l], \tilde{x}_0)]$ lifting l. This shows that $[(f, \tilde{x}_0)][l] = [(f[l], \tilde{x}_0)]$, so that the bijection is indeed a map of $\pi_1(X, x)$-sets. □

Proposition 6.11. ***Fib*** *and* ***Bor*** *restrict to an equivalence of categories*

$$\textbf{\textit{PathConnCov}}_*(X,x) \simeq \textbf{\textit{Trans-}}\pi_1(X,x)\textbf{\textit{-Set}}_*.$$

Proof. Suppose $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering map, that $\tilde{X}$ is path-connected, and let f, f' be in the fiber $\pi^{-1}(x)$. We can choose a path p from f to f'. Then $l = \pi \circ p$ is a loop in X and we have $f[l] = f'$, so the action of $\pi_1(X, x)$ on $\pi^{-1}(x)$ is transitive.

Conversely, suppose (X, x) is a based space, that F is a transitive $\pi_1(X, x)$-set and construct

$$\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$$

as in definition 6.9. Given two points in $\tilde{X}$, because the $\pi_1(X, x)$-action on F is transtive, we can choose their representatives so that the F-coordinates are the same; we can then connect them by a path because $\tilde{X}_0$ is path-connected. Thus, $\tilde{X}$ is path-connected. ☐

Recall from elementary group theory that transitive G-sets with a distinguished point are determined up to isomorphism by the stabilizer of of that disinguished point.

Theorem 6.12. *We have an equivalence of categories*

$$\begin{array}{ccc} \textbf{\textit{PathConnCov}}_*(X,x) & \longleftrightarrow & \{\textit{subgroups of } \pi_1(X,x)\} \\ \Big(\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)\Big) & \longmapsto & \operatorname{im}\Big(\pi_* : \pi_1(\tilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x)\Big) \\ H\backslash\tilde{X}_0 & \longleftarrow\!\shortmid & H. \end{array}$$

Proof. Suppose $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering map. Under the action of $\pi_1(X, x)$ on $\pi^{-1}(x)$, the stabilizer of $\tilde{x}$ consists of the loops in X starting at x which lift to loops in $\tilde{X}$ starting at $\tilde{x}$. This is precisely

$$\operatorname{im}\Big(\pi_* : \pi_1(\tilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x)\Big).$$

Also, $\Big(H\backslash\pi_1(X, x_0)\Big) \times \tilde{X}_0 \Big/ \sim\; = H\backslash\tilde{X}_0.$ ☐

Corollary 6.13. *Let $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ be a covering space in which $\tilde{X}$ and X are path-connected. The number of sheets of the covering equals the index of $\pi_*(\pi_1(\tilde{X}, \tilde{x}))$ in $\pi_1(X, x)$.*

Remark 6.14. Forgetting the basepoint in a covering space of (X, x) causes us to forget about the distinguished point in the corresponding $\pi_1(X, x)$-set. This means that the corresponding stabilizer is only defined up to conjugacy class.

The geometric reason for this is as follows. Suppose (X, x) is a based space, $\pi : \tilde{X} \longrightarrow X$ is a covering space, and $\tilde{x}_1, \tilde{x}_2 \in \pi^{-1}(x)$. If p is a path from $\tilde{x}_1$ to $\tilde{x}_2$, then $l = \pi p$ is a loop based at x, and the following diagram commutes. Thus, changing the basepoint in a path-connected covering space corresponds to changing the image subgroup to a conjugate subgroup.

$$\begin{array}{ccc} \pi_1(\tilde{X}, \tilde{x}_1) & \overset{\beta_h}{\underset{\cong}{\longleftarrow}} & \pi_1(\tilde{X}, \tilde{x}_2) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \pi_1(X, x) & \overset{c_{[l]}}{\underset{\cong}{\longleftarrow}} & \pi_1(X, x) \end{array}$$

Corollary 6.15. *We have an equivalence of categories*

$$\textbf{\textit{PathConnCov}}(X,x) \longleftrightarrow \{\textit{conjugacy classes of subgroups of } \pi_1(X,x)\}.$$

Definition 6.16. Let $\pi : \tilde{X} \longrightarrow X$ be a covering map. The automorphisms of a covering π are called *deck transformations*. The group of deck transformations $\mathrm{Aut}(\pi)$ acts on $\tilde{X}$ on the left. π is called *normal* if for all $x \in X$, the action of $\mathrm{Aut}(\pi)$ on $\pi^{-1}(x)$ is transitive.

Proposition 6.17. *Suppose (X,x) is a space, that $\pi : \tilde{X} \to X$ is a path-connected covering space, $\tilde{x}_1, \tilde{x}_2 \in \pi^{-1}(x)$, p is a path from $\tilde{x}_1$ to $\tilde{x}_2$, and $l = \pi p$. Then there is a deck transformation taking $\tilde{x}_1$ to $\tilde{x_2}$ if and only if conjugating by $[l]$ is trivial on $\pi_*(\pi_1(\tilde{X}, \tilde{x}_1))$.*

Proof. There is a deck transformation taking $\tilde{x}_1$ to $\tilde{x_2}$ if and only if $\pi_*(\pi_1(\tilde{X}, \tilde{x}_1)) = \pi_*(\pi_1(\tilde{X}, \tilde{x}_2))$ and the previous remark tells us that these groups are related by conjugation by $[l]$. □

Proposition 6.18. *Suppose (X,x) is a space, that $\pi : \tilde{X} \to X$ is a path-connected covering space, and $\tilde{x} \in \pi^{-1}(x)$. Let $H = \pi_*(\pi_1(\tilde{X}, \tilde{x}))$.*

π is a normal covering if and only H is a normal subgroup of $\pi_1(X,x)$.

In any case, we have an isomorphism $N(H)/H \longrightarrow \mathrm{Aut}(\pi)$, where $N(H)$ denotes the normalizer of H.

Proof. The claim about normality follows from the previous proposition.

First note that by connectedness, a deck transformation is determined by the image of $\tilde{x}$. Define $\Phi : N(H) \to \mathrm{Aut}(\pi)$ by $\Phi[l](\tilde{x}) = \tilde{x}[l]$, that is, given $[l] \in N(H)$, lift l to a path p which starts at $\tilde{x}$, and let $\Phi[l]$ be the deck transformation which takes $\tilde{x}$ to $p(1)$. The first sentence of this paragraph together with the previous proposition ensure that Φ is well-defined and surjective.

We check Φ is a homomorphism. Suppose $[l_1], [l_2] \in \pi_1(X,x)$ and that $p_1 : (I,0) \to (X, \tilde{x})$ and $p_2 : (I,0) \to (X, \tilde{x})$ lift l_1 and l_2, respectively. We have $\Phi[l_1](\tilde{x}) = p_1(1)$ and $\Phi[l_2](\tilde{x}) = p_2(1)$. So $\Phi[l_1]p_2 : (I,0) \longrightarrow (\tilde{X}, p_1(1))$ lifts l_2, and $p_1 \cdot (\Phi[l_1]p_2)$ lifts $l_1 \cdot l_2$. Thus,

$$\Phi[l_1 \cdot l_2](\tilde{x}) = (p_1 \cdot (\Phi[l_1]p_2))(1) = (\Phi[l_1]p_2)(1) = \Phi[l_1](p_2(1)) = \Phi[l_1](\Phi[l_2](\tilde{x})) = \Phi[l_1]\Phi[l_2](\tilde{x})$$

which implies $\Phi[l_1 \cdot l_2] = \Phi[l_1]\Phi[l_2]$. Finally, $\ker \Phi = \mathrm{stab}(\tilde{x}) = H$. □

A covering space proof of theorem 4.10. Suppose (X, x_0) is a based connected CW-complex.

Suppose $p : I \to X$ is a path with $p(0) = x_0$, and $\varphi : (S^1, *) \longrightarrow (X, p(1))$. Let Y be the space obtained by attaching a 2-cell to X using φ, let l be the loop corresponding to φ, and let N be the normal subgroup generated by $[p \cdot l \cdot \overline{p}]$.

Let $\pi : \tilde{X} \longrightarrow X$ be the covering space corresponding to N. Because N is normal, $\varphi : (S^1, *) \longrightarrow (X, p(1))$ has lifts to $\tilde{X}$ indexed by G/N. Attach a 2-cell for each element of G/N to get a covering $\tilde{Y} \longrightarrow Y$.

It is clear that N is in the kernel of $\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$. Conversely, suppose $[l]$ is in the kernel of $\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$. This means we have a nullhomotopy of l in Y. This means that l lifts to a loop in $\tilde{Y}$, and hence in $\tilde{X}$. This means $[l]$ is in the image of the map induced by π, i.e. it is in N. □

Example 6.19. Consider $X = S^1 \vee S^1$. Then $\pi_1(X, *)$ is a free group on two generators σ and τ. We have $\pi_1(X, *)$-sets:

$E = \{*\}$.

$\Sigma = \{l_\Sigma, r_\Sigma\}$, σ acts trivially, and τ acts non-trivially.

$T = \{c_T, l_T, r_T\}$, $\sigma : c_T \longmapsto l_T \longmapsto r_T \longmapsto c_T$, τ swaps l_T and r_T and fixes c_T.

$G = \{1, 1', 2, 2', 3, 3'\}$, τ swaps prime and unprimed, $\sigma = (123)(1'2'3')^{-1}$.

Taking stabilizers:

$\mathrm{Stab}(*) \supset \langle \sigma, \tau \rangle$.

$\mathrm{Stab}(l_\Sigma), \mathrm{Stab}(r_\Sigma) \supset \langle \tau^2, \sigma, \tau^{-1}\sigma\tau \rangle$.

$\mathrm{Stab}(c_T) \supset \langle \sigma^3, \tau, \sigma\tau\sigma, \sigma\tau^{-1}\sigma \rangle$.

$\mathrm{Stab}(l_T) \supset \langle \sigma^3, \sigma^{-1}\tau\sigma, \tau\sigma^2, \tau^{-1}\sigma^2 \rangle$.

$\mathrm{Stab}(r_T) \supset \langle \sigma^3, \sigma\tau\sigma^{-1}, \sigma^2\tau, \sigma^2\tau^{-1} \rangle$.

$\mathrm{Stab}(1), \mathrm{Stab}(1'), \ldots, \mathrm{Stab}(3), \mathrm{Stab}(3') \supset \langle \sigma^3, \tau^2, \sigma^{-1}\tau^2\sigma, \sigma\tau^2\sigma^{-1}, \sigma\tau\sigma\tau, \tau\sigma\tau\sigma, \sigma\tau\sigma^{-1}\tau\sigma \rangle$.

By constructing covers you find that we have equality and that the groups are generated freely.

Attaching 2-cells to X according to the words σ^3, τ^2, $\sigma\tau\sigma\tau$, we obtain a CW-complex Y with $\pi_1(Y, *) = \langle \sigma, \tau | \sigma^3, \tau^2, \sigma\tau\sigma\tau \rangle \cong \Sigma_3$. The sets above are also Σ_3-sets; we obtain the complete Galois correspondence for Y and Σ_3.

7 Hawaiian Earring

Let $S^1_n = \{x \in \mathbb{R}^2 : \ \|x - \frac{1}{n}\| = \frac{1}{n}\}$ and $\mathbb{H} = \bigcup_{n=1}^{\infty} S^1_n$. We will calculate $\pi_1(\mathbb{H}, 0)$.

Notice that for $N \in \mathbb{N}$, $\bigcup_{n=1}^{N} S^1_n \cong \bigvee_{n=1}^{N} S^1$. We will use this identification throughout.

We have retractions $r_N : \mathbb{H} \longrightarrow \bigvee_{n=1}^{N} S^1$. Moreover, these retractions are compatible with the collapse maps $c_N : \bigvee_{n=1}^{N+1} S^1 \longrightarrow \bigvee_{n=1}^{N} S^1$, that is the following square commutes for all $N \in \mathbb{N}$:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{r_{N+1}} & \bigvee_{n=1}^{N+1} S^1 \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle c_N} \\ \mathbb{H} & \xrightarrow{r_N} & \bigvee_{n=1}^{N} S^1. \end{array}$$

Thus, these retractions give compatible surjections $(r_N)_* : \pi_1(\mathbb{H}, 0) \longrightarrow \pi_1(\bigvee_{n=1}^{N} S^1, *)$.

We obtain a map $\{(r_N)_*\} : \pi_1(\mathbb{H}, 0) \longrightarrow \lim_N \pi_1(\bigvee_{n=1}^{N} S^1, *)$.

In case you're not familiar with (inverse) limits:

$$\lim_N \ \pi_1\left(\bigvee_{n=1}^{N} S^1, *\right) = \left\{\{g_N\}_{N=1}^{\infty} : \ \forall N \in \mathbb{N}, \ g_N \in \pi_1\left(\bigvee_{n=1}^{N} S^1, *\right), \ (c_N)_*(g_{N+1}) = g_N\right\}.$$

Let $a_1, a_2, a_3, \ldots$ be chosen so that for each $N \in \mathbb{N}$, $\pi_1(\bigvee_{n=1}^{N} S^1, *)$ is a free group on $a_1, \ldots, a_N$. Given $\{g_N\}_{N=1}^{\infty} \in \lim_N \pi_1(\bigvee_{n=1}^{N} S^1, *)$, we can make sure that each g_N is a reduced word. We can then write $d_n(g_N)$ for the number of times that $a_n^{\pm 1}$ appears in g_N. Let

$$H = \left\{\{g_N\}_{N=1}^{\infty} \in \lim_N \ \pi_1\left(\bigvee_{n=1}^{N} S^1, *\right) : \ \forall n \in \mathbb{N}, \ \lim_{N \to \infty} d_n(g_N) \text{ is finite}\right\}.$$

We claim that $\{(r_N)_*\}$ is an injection with image H.

- $\mathrm{im}\{(r_N)_*\} \subseteq H$.

  Suppose $l : (I, \partial I) \longrightarrow (\mathbb{H}, 0)$ is a loop in $\mathbb{H}$.

  l is uniformly continuous, so for each $n \in \mathbb{N}$, we can find a $\delta_n > 0$ such that

  $$\forall s \in I, \ \forall t \in I, \ |s - t| < \delta_n \implies \|l(s) - l(t)\| < \frac{1}{n}.$$

  Thus, $\lim_{N \to \infty} d_n((r_N)_*[l]) \leq \frac{1}{\delta_n}$, and $\{(r_N)_*\}[l] = \{(r_N)_*[l]\} \in H$.

  As an example, let $h_n = a_1 a_n$ and $g_N = h_1 h_2 h_3 \ldots h_{N-1} h_N a_1^{-N}$. Then it should be clear that $\{g_N\}$ is not in the image of $\{(r_N)_*\}$.

- $H \subseteq \mathrm{im}\{(r_N)_*\}$.

  Suppose $\{g_N\} \in H$. Choose an increasing sequence of natural numbers $(N_k)_{k=1}^{\infty}$ such that

  $$d_k(g_{N_k}) = \lim_{N \to \infty} d_k(g_N).$$

  Chop the interval I into $2N_1 + 1$ pieces. Chop the 1-st, 3-rd, 5-th, ..., $(2N_1 + 1)$-st pieces into $2N_2 + 1$ pieces. Chop the 1-st, 3-rd, ..., $(2N_2 + 1)$-st of each of these pieces into $2N_3 + 1$ pieces. Keep going.

We now define a loop $l : (I, \partial I) \to (\mathbb{H}, 0)$ as follows. Use g_{N_1} to define what happpens on the 2-nd, 4-th, ..., $2N_1$-th piece of the interval in the first level of chopping: either do a_1 or a_1^{-1}. Use the elements g_{N_2} to define things on the even pieces of next level of chopping. I chopped more than necessary and there will be some constant paths. Keep going to get a loop l with $\{(r_N)_*\}[l] = \{g_N\}$.

The idea above was simply to put in the correct amount of looping around the k-th circle S_k^1, while leaving space to do the loops around smaller circles. Checking continuity of the loop using ϵ-δ would make explicit use of the sequence $(N_k)_{k=1}^{\infty}$.

- $\ker\{(r_N)_*\} = 1$.

  Suppose $l : (I, \partial I) \longrightarrow (\mathbb{H}, 0)$ is a loop in $\mathbb{H}$ and that $\{(r_N)_*\}[l] = 1$.

  We must show $[l] = 1$. This is difficult.

8 The prism proof

$\Delta^n \times I$ is built from $(n+1)$-simplices.

Label the vertices of $\Delta^n \times 0 = [v_0, \ldots, v_n]$ and $\Delta^n \times 1 = [w_0, \ldots, w_n]$.

Let $\Gamma[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, v_i, w_i, \ldots, w_n]$. Then

$$\begin{aligned}
\partial\Gamma[v_0, \ldots, v_n] &= \sum_{i=0}^{n} (-1)^i \partial[v_0, \ldots, v_i, w_i, \ldots, w_n] \\
&= \sum_{i=0}^{n} (-1)^i \Bigg[\sum_{j=0}^{i} (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n] + \\
&\qquad \sum_{j=i}^{n} (-1)^{j+1} [v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j \ldots, w_n] \Bigg]
\end{aligned}$$

$$\begin{aligned}
\Gamma\partial[v_0, \ldots, v_n] &= \sum_{j=0}^{n} (-1)^j \Gamma[v_0, \ldots, \hat{v}_j, \ldots, v_n] \\
&= \sum_{j=0}^{n} (-1)^j \Bigg[\sum_{i=0}^{j-1} (-1)^i [v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n] + \\
&\qquad \sum_{i=j+1}^{n} (-1)^{i-1} [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i \ldots, w_n] \Bigg]
\end{aligned}$$

and

$$\begin{aligned}
(\partial\Gamma + \Gamma\partial)[v_0, \ldots, v_n] &= \sum_{i=0}^{n} (-1)^i \Bigg[(-1)^i [v_0, \ldots, v_{i-1}, w_i, \ldots, w_n] + \\
&\qquad (-1)^{i+1} [v_0, \ldots, v_i, w_{i+1}, \ldots, w_n] \Bigg] \\
&= \sum_{i=0}^{n} \Bigg[[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n] - [v_0, \ldots, v_i, w_{i+1}, \ldots, w_n] \Bigg] \\
&= [w_0, \ldots, w_n] - [v_0, \ldots, v_n].
\end{aligned}$$

Thus, $h : C_n(X) \longrightarrow C_{n+1}(X \times I)$, $\sigma \longmapsto (\sigma \times 1)_\#(\Gamma)$ defines a chain homotopy from $(i_0)_\#$ to $(i_1)_\#$ where $i_0, i_1 : X \longrightarrow X \times I$ are the inclusions $X \longmapsto (x, 0)$ and $x \longmapsto (x, 1)$.

9 Poincaré Duality

Let M be an oriented smooth manifold without boundary, which is not necessarily compact. Let us also assume that M has the structure of a Δ-complex in which the characteristic maps are smooth, and that the homology of M is finitely generated.

We have inclusions of chain complexes $C_*^{\Delta}(M) \longrightarrow C_*^{\text{sm}}(M) \longrightarrow C_*^{\text{sing}}(M)$. The whole composite induces an isomorphism on homology. One can use a relative version of smooth approximation to show each individual map induces an isomorphism on homology, and we can use homology based on smooth chains in place of singular homology.

We have a homomorphism $\rho_k : \Omega^k(M) \longrightarrow \text{Hom}(C_k^{\text{sm}}(M), \mathbb{R})$ defined by

$$\rho_k(\omega)(\sigma) = \int_{\Delta^k} \sigma^* \omega.$$

Moreover, by Stokes' theorem

$$\rho_{k+1}(d\omega)(\sigma) = \int_{\Delta^{k+1}} \sigma^*(d\omega) = \int_{\Delta^{k+1}} d(\sigma^*\omega) = \int_{\partial\Delta^{k+1}} \sigma^*\omega = \rho_k(\omega)(\partial\sigma) = \delta(\rho_k(\omega))(\sigma)$$

so ρ_k induces $H^k_{DR}(M) \longrightarrow H^k(M;\mathbb{R})$. De Rham's theorem says this is an isomorphism.

Moreover, the UCT gives an isomorphism $H^k(M;\mathbb{R}) \longrightarrow \text{Hom}(H_k(M;\mathbb{R}), \mathbb{R})$.

Thus, we have an isomorphism

$$H^k_{DR}(M) \longrightarrow \text{Hom}(H_k(M;\mathbb{R}), \mathbb{R}), \ [\omega] \longmapsto \left(\left[\sum_i x_i \sigma_i \right] \longmapsto \sum_i x_i \int_{\Delta^k} \sigma_i^* \omega \right).$$

By our finiteness assumption, we also have an isomorphism

$$H_k(M;\mathbb{R}) \longrightarrow \text{Hom}(H^k_{DR}(M), \mathbb{R}), \ \left[\sum_i x_i \sigma_i \right] \longmapsto \left([\omega] \longmapsto \sum_i x_i \int_{\Delta^k} \sigma_i^* \omega \right).$$

Poincaré duality in the smooth setting tells us that the map

$$PD_{\text{sm}} : H^k_{DR}(M) \longrightarrow \text{Hom}(H^{n-k}_{DR,c}(M), \mathbb{R}), \ [\omega] \longmapsto \left([\eta] \longmapsto \int_M \eta \wedge \omega \right)$$

is an isomorphism. By our finiteness assumption, we also have an isomorphism

$$H^{n-k}_{DR,c}(M) \longrightarrow \text{Hom}(H^k_{DR}(M), \mathbb{R}), \ [\eta] \longmapsto \left([\omega] \longmapsto \int_M \eta \wedge \omega \right)$$

is an isomorphism.

Thus, we have an isomorphism

$$H_k(M;\mathbb{R}) \longrightarrow \text{Hom}(H^k_{DR}(M), \mathbb{R}) \longleftarrow H^{n-k}_{DR,c}(M).$$

$\left[\sum_i x_i \sigma_i \right]$ corresponds to an element $[\eta]$ such that

$$\sum_i x_i \int_{\Delta^k} \sigma_i^* \omega = \int_M \eta \wedge \omega$$

for all $\omega \in \Omega^k(M)$. In fact, such an isomorphism exists without a finiteness assumption. Moreover, in the case that $\sum_i x_i\sigma_i$ is a submanifold S we can describe $[\eta]$ nicely. Let T be a tublar neighborhood of S so that T is isomorphic to the normal bundle of S in M. We can take η to be a form which is a bump $(n-k)$-form on each fiber $\mathbb{R}^{n-k}$.

Now let M be a compact manifold without boundary. Let us also assume that M has the structure of a Δ-complex. We'll describe an isomorphism $H^{n-k}(M;\mathbb{Z}/2) \longrightarrow H_k(M;\mathbb{Z}/2)$. When M is oriented we'll describe an isomorphism $H^{n-k}(M) \longrightarrow H_k(M)$. This map will be chosen so that

$$H^k(M;R) \longrightarrow \mathrm{Hom}(H_k(M;R),R) \longrightarrow \mathrm{Hom}(H^{n-k}(M;R),R)$$

is closely related to the cup product just like the map PD_{sm}.

10 Final

10.1 Questions

1. (a) As concisely as possible, while giving essential details, describe the definition of cellular homology for CW-complexes and why it is isomorphic to singular homology.

   (b) Calculate, as groups, the homology of $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ with $\mathbb{Z}$ and $\mathbb{Z}/2$-coefficients, carefully justifying any degree calculations.

   (c) Use the UCT to calculate, as groups, the cohomology of $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ with $\mathbb{Z}$ and $\mathbb{Z}/2$-coefficients. Describe the natural maps

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}) \longrightarrow H^*(\mathbb{RP}^\infty;\mathbb{Z}/2),\ H^*(\mathbb{CP}^\infty;\mathbb{Z}) \longrightarrow H^*(\mathbb{CP}^\infty;\mathbb{Z}/2).$$

   (d) Explain how the UCT and Poincaré duality can be used to calculate the rings

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}/2),\ H^*(\mathbb{CP}^\infty;\mathbb{Z}).$$

   (e) Calculate the rings $H^*(\mathbb{RP}^\infty;\mathbb{Z})$ and $H^*(\mathbb{CP}^\infty;\mathbb{Z}/2)$.

   (f) Let $q : \mathbb{RP}^\infty \longrightarrow \mathbb{CP}^\infty$ be the natural quotient map obtained by regarding both spaces as quotients of S^∞.

   Describe the square

$$\begin{array}{ccc} H^*(\mathbb{RP}^\infty;\mathbb{Z}) & \xleftarrow{\ q^*\ } & H^*(\mathbb{CP}^\infty;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) & \xleftarrow{\ q^*\ } & H^*(\mathbb{CP}^\infty;\mathbb{Z}/2) \end{array}$$

   Describe $q_* : H_*(\mathbb{RP}^\infty;\mathbb{Z}) \longrightarrow H_*(\mathbb{CP}^\infty;\mathbb{Z})$ and $q_* : H_*(\mathbb{RP}^\infty;\mathbb{Z}/2) \longrightarrow H_*(\mathbb{CP}^\infty;\mathbb{Z}/2)$.

   (g) $\mathbb{CP}^n$ can be regarded as polynomials of degree $\leq n$ modulo multiplication by a non-zero complex number. Multiplication of polynomials determines a map

$$\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty.$$

   Calculate $\mu_* : H_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty;\mathbb{Z}) \longrightarrow H_*(\mathbb{CP}^\infty;\mathbb{Z})$ justifying everything carefully.

   (h) Similarly, there is a map $\mu : \mathbb{RP}^\infty \times \mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty$.

   Calculate $\mu_* : H_*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty;\mathbb{Z}/2) \longrightarrow H_*(\mathbb{RP}^\infty;\mathbb{Z}/2)$ justifying everything carefully.

2. (a) The homology $H_*(X;\mathbb{Z})$ of a space X is a graded abelian group.
Which graded abelian groups arise as the homology of a space? Prove your claim.

(b) Suppose M is a simply-connected closed 4-dimensional manifold (compact without boundary), and that

$$\text{rank } H_2(M;\mathbb{Z}) \leq 2.$$

Based on this information, describe the possibilities for the cohomology ring $H^*(M;\mathbb{Z})$, and find an example of M realizing each.

10.2 Solutions

1. (a) I'd copy pages 4 and 5 of
`http://math.ucla.edu/~mjandr/Math225C/hand_lectures.pdf`.
I'm a little sad that you all opted to copy Hatcher instead.

(b) The $\mathbb{CP}^\infty$ calculations are straightforward because the non-trivial parts of the skeletal filtration are given by

$$* = \mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \mathbb{CP}^3 \subset \dots$$

and $\mathbb{CP}^n/\mathbb{CP}^{n-1} = S^{2n}$. In particular, the odd-dimensional cellular groups are 0.
The skeletal filtration of $\mathbb{RP}^\infty$ is given by

$$* = \mathbb{RP}^0 \subset \mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \mathbb{RP}^3 \subset \dots$$

and $\mathbb{RP}^n/\mathbb{RP}^{n-1} = S^n$. To understand the cellular chain complex, one needs to understand the degree of the map $f : S^n \longrightarrow \mathbb{RP}^n \longrightarrow \mathbb{RP}^n/\mathbb{RP}^{n-1} = S^n$. Pick $x \in S^n - S^{n-1}$, and let y be its image. The preimage of y is $\{-x, x\}$. Near $-x$ and x, f is a homeomorphism. These homeomorphisms agree up to composition with the antipodal map which has degree $(-1)^{n+1}$. So, when n is even, f has degree 0, and when f is odd, f has degree 2.

(c) I did it.

(d) The UCT together with Poincaré duality give isomorphisms

$$H^k(\mathbb{RP}^n;\mathbb{Z}/2) \longrightarrow \text{Hom}(H_k(\mathbb{RP}^n;\mathbb{Z}/2),\mathbb{Z}/2) \longrightarrow \text{Hom}(H^{n-k}(\mathbb{RP}^n;\mathbb{Z}/2),\mathbb{Z}/2).$$

$$H^{2k}(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow \text{Hom}(H_{2k}(\mathbb{CP}^n;\mathbb{Z}),\mathbb{Z}) \longrightarrow \text{Hom}(H^{2n-2k}(\mathbb{CP}^n;\mathbb{Z}),\mathbb{Z}).$$

This tells us that the cup products

$$H^k(\mathbb{RP}^n;\mathbb{Z}/2) \otimes H^{n-k}(\mathbb{RP}^n;\mathbb{Z}/2) \longrightarrow H^n(\mathbb{RP}^n;\mathbb{Z}/2)$$

$$H^{2k}(\mathbb{CP}^n;\mathbb{Z}) \otimes H^{2n-2k}(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow H^{2n}(\mathbb{CP}^n;\mathbb{Z})$$

are perfect pairings.
Thus, the cup product structures on $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$ and $H^*(\mathbb{CP}^n;\mathbb{Z})$ are as non-trivial as possible. Using the quotient maps,

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) \longrightarrow H^*(\mathbb{RP}^n;\mathbb{Z}/2),\ H^*(\mathbb{CP}^\infty;\mathbb{Z}) \longrightarrow H^*(\mathbb{CP}^n;\mathbb{Z}),$$

we conclude that $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[w]$ and $H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[c]$ where $|w| = 1$ and $|c| = 2$.

(e) From part (c), it is immediate that

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}) = \mathbb{Z}[w^2]/(2w^2),\;\; H^*(\mathbb{CP}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[c].$$

(f) Let $q : \mathbb{RP}^\infty \longrightarrow \mathbb{CP}^\infty$ be the natural quotient. The square

$$\begin{array}{ccc} \mathbb{Z}[w^2]/(2w^2) & \xleftarrow{\;q^*\;} & \mathbb{Z}[c] \\ \downarrow & & \downarrow \\ \mathbb{Z}/2[w] & \xleftarrow{\;q^*\;} & \mathbb{Z}/2[c] \end{array}$$

is determined by the bottom map, which is determined by where c is mapped to.
q restricts to a map $q| : \mathbb{RP}^2 = S^2/\pm 1 \longrightarrow S^3/S^1 = \mathbb{CP}^1$. Let $x \in \mathbb{RP}^2 - \mathbb{RP}^1$, and let y be its image under q. Then the pre-image of y is $\{x\}$, and $q|$ is a local homeomorphism. We conclude that c is mapped to w^2.
$q_* : H_*(\mathbb{RP}^\infty;\mathbb{Z}) \longrightarrow H_*(\mathbb{CP}^\infty;\mathbb{Z})$ is the only map it can be.
Since it's the dual of q^*, $q_* : H_*(\mathbb{RP}^\infty;\mathbb{Z}/2) \to H_*(\mathbb{CP}^\infty;\mathbb{Z}/2)$ is as non-trivial as possible.

(g) Let $\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty$ be given by polynomial multiplication.
The following diagram commutes

$$\begin{array}{ccccc} \mathbb{CP}^0 \times \mathbb{CP}^\infty & \longrightarrow & \mathbb{CP}^\infty \times \mathbb{CP}^\infty & \longleftarrow & \mathbb{CP}^\infty \times \mathbb{CP}^0 \\ & \underset{\cong}{\searrow} & \downarrow \mu & \underset{\cong}{\swarrow} & \\ & & \mathbb{CP}^\infty & & \end{array}$$

We have $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[c]$, and Künneth says that $H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = \mathbb{Z}[c] \otimes \mathbb{Z}[c]$. The diagram just drawn gives $\mu^*(c) = 1 \otimes c + c \otimes 1$. The binomial theorem says that

$$\mu^*(c^n) = \sum_{i+j=n} \binom{n}{i} c^i \otimes c^j.$$

μ_* is dual to μ^*: it takes $e_{2i} \otimes e_{2j}$ to $\frac{(i+j)!}{i!j!} e_{2i+2j}$.

(h) Let $\mu : \mathbb{RP}^\infty \times \mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty$ be given by polynomial multiplication.
Then $\mu_* : H_*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty;\mathbb{Z}/2) \longrightarrow H_*(\mathbb{RP}^\infty;\mathbb{Z}/2)$ takes $e_i \otimes e_j$ to $\frac{(i+j)!}{i!j!} e_{i+j}$.

2. (a) The homology $H_*(X;\mathbb{Z})$ of a space X is free abelian in dimension 0 and of rank greater than or equal to 1. In dimensions bigger than 1 it can be any abelian group.
Let G be such a graded abelian group and let $n \in \{1, 2, 3, \ldots\}$. First, find a CW-complex X_n such that $\tilde{H}_n(X) = G_n$ and $\tilde{H}_m(X_n) = 0$ when $m \neq n$. Let $X = \bigvee_{n\in\{1,2,3,\ldots\}} X_n$. If necessary, add some disjoint points to X. To find the CW-complexes X_n, just do what you did for fundamental groups in higher-dimensions.

(b) Suppose M is a simply-connected closed 4-dimensional manifold (compact without boundary), and that rank $H_2(M;\mathbb{Z}) \leq 2$. Since M is simply-connected, M is orientable and Poincaré duality applies. Together with the UCT, we obtain

$$H_1(M) = H^1(M) = H_3(M) = H^3(M) = 0,$$

$$H_0(M) \cong H^0(M) \cong H_4(M) \cong H^4(M) \cong \mathbb{Z}.$$

We also obtain $H_2(M) \cong H^2(M) \cong \mathbb{Z}^r$ where $r = 0, 1$, or 2.
When $r = 0$, the cohomology ring is determined, and S^4 realizes it.
When $r = 1$, Poicaré duality forces the cohomology ring to be $\mathbb{Z}/[c]/(c^3)$ which is realized by $\mathbb{CP}^2$.
When $r = 2$, Poincaré duality gives a perfect pairing $H^2(M) \otimes H^2(M) \to H^4(M)$. We claim that the only symmetric perfect pairings $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ up to isomorphism are given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see these are realized by $S^2 \times S^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. $\overline{\mathbb{CP}}^2$ is $\mathbb{CP}^2$ with the opposite orientation, which means that when we do the connected sum the gluing is different.

To show the perfect pairings are as described comes down to linear algebra over $\mathbb{Z}$. Let α be an element which can be extended to a basis for $H^2(M)$. There are two cases:

i. α^2 generates $H^4(M)$.
ii. We can find a $\beta \in H^2(M) - \mathbb{Z}\langle\alpha\rangle$ such that $\alpha\beta$ generates $H^4(M)$.

We deal with these in turn:

i. The matrix of the cup product with respect to a basis $\alpha, \beta \in H^2(M)$ and $\gamma \in H^4(M)$ looks like $\begin{pmatrix} 1 & b \\ b & c \end{pmatrix}$, with $c - b^2 = \pm 1$.
Since $x^2 + 2bxy + cy^2 = x^2 + 2bxy + (b^2 \pm 1)y^2 = (x + by)^2 \pm y^2$, by changing basis, we can convert the matrix to $\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.
ii. The matrix of the cup product with respect to a basis $\alpha, \beta \in H^2(M)$ and $\gamma \in H^4(M)$ looks like $\begin{pmatrix} a & 1 \\ 1 & d \end{pmatrix}$, with $ad - 1 = \pm 1$. There are two cases.
   A. $ad = 0$. Without loss of generality, $d = 0$.
   When a is even, $ax^2 + 2xy = 2x(\frac{a}{2}x + y)$ shows by changing basis, we can convert the matrix to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   When a is odd, $ax^2 + 2xy = (\frac{a+1}{2}x + y)^2 - (\frac{a-1}{2}x + y)^2$ shows by changing basis, we can convert the matrix to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
   B. $ad = 2$. Without loss of generality, $a = \pm 1$, $d = \pm 2$.
   Since $\pm x^2 + 2xy \pm 2y^2 = \pm(x \pm y)^2 \pm y^2$, by changing basis, we can convert the matrix to $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. By changing γ, if necessary, we can convert it to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Notice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are not isomorphic, even though they are isomorphic over the reals.