LAST TIME

$$\tilde{H}_n(X) := \ker\big(H_n(X) \to H_n(*)\big) \xrightarrow{\cong} H_n(X,*)$$

$$= H_n(X) \quad \text{IF } n > 0.$$

GIVEN $A \subset X$ HAVE LESs

$$\begin{tikzcd}
H_{n+1}(X,A) \arrow[r, "\delta"] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r, "\delta"] & H_{n-1}(A) \\
H_{n+1}(X,A) \arrow[u, "="] \arrow[r] \arrow[d, "q_*"] & \tilde{H}_n(A) \arrow[u] \arrow[r] & \tilde{H}_n(X) \arrow[u] \arrow[r] \arrow[ddr, "q_*"'] & H_n(X,A) \arrow[u, "="] \arrow[r] \arrow[d, "q_*"] & \tilde{H}_{n-1}(A) \arrow[u] \\
H_{n+1}\left(X/A, A/A\right) & & & H_n\left(X/A, A/A\right) & \\
\tilde{H}_{n+1}\left(X/A\right) \arrow[u, "\cong"] & & & \tilde{H}_n\left(X/A\right) \arrow[u, "\cong"] &
\end{tikzcd}$$

THIS TIME: $\quad q_* : H_n(X,A) \longrightarrow H_n\left(X/A, A/A\right)$

IS AN ISO. IF (X,A) IS GOOD.

FIRST THOUGH: RECALL WE SHOWED

$$\tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k. \end{cases}$$

WHAT'S THE GENERATOR FOR $\tilde{H}_n(S^n)$?

LET

$$\begin{tikzcd}
\sigma_{S^{n-1}} : \Delta^{n-1} \arrow[r] \arrow[d] & \Delta^{n-1}/\partial\Delta^{n-1} \arrow[d, "="'] \arrow[r, phantom, "\cong"] & S^{n-1} \arrow[d, "="] \\
\partial\Delta^{n} \arrow[r, "\cong"] \arrow[d] & \partial\Delta^{n}/\Lambda^{n-1} \arrow[d] \arrow[r, phantom, "\cong"] & S^{n-1} \arrow[d, hook] \\
\sigma_{D^n} : \Delta^{n} \arrow[r, "\cong"] \arrow[d, "="'] & \Delta^{n}/\Lambda^{n-1} \arrow[d] \arrow[r, phantom, "\cong"] & D^{n} \arrow[d] \\
\sigma_{S^n} : \Delta^{n} \arrow[r] & \Delta^{n}/\partial\Delta^{n} \arrow[r, phantom, "\cong"] & S^{n}
\end{tikzcd}$$

WHERE $\Delta^n = [v_0, v_1, \dots, v_n]$

$\Delta^{n-1} = [v_0, v_1, \dots, v_{n-1}]$

$\Lambda^{n-1} = \bigcup_{i<n} [v_0, \dots, \hat{v}_i, \dots, v_n]$.

LET $x_n \in \tilde{H}_n(S^n)$ BE DEFINED BY $[\sigma_{S^n} - *]$.

$$\begin{array}{ccccccc}
\tilde{H}_n(S^n) & \xrightarrow{\cong} & H_n(S^n, *) & \xleftarrow{\cong} & H_n(D^n, S^{n-1}) & \xrightarrow{\cong} & \tilde{H}_{n-1}(S^{n-1}) \\
x_n & \longmapsto & [\sigma_{S^n}] & \longmapsfrom & [\sigma_{D^n}] & \longmapsto & (-1)^n x_{n-1}.
\end{array}$$

x_0 GENERATES $\tilde{H}_0(S^0)$, SO x_n GENERATES $\tilde{H}_n(S^n)$.

ALSO $(\sigma_{D^n})_* : H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_n(D^n, S^{n-1})$, $[1_{\Delta^n}] \mapsto [\sigma_{D^n}]$

SO $[1_{\Delta^n}]$ GENERATES $H_n(\Delta^n, \partial\Delta^n)$.

SUPPOSE X IS A CW-COMPLEX.

HOW TO CALCULATE $H_*(X)$?

HAVE SKELETAL FILTRATION

$$\phi = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \cdots \subset X.$$

CAN PUT ALL LESs TOGETHER :

$$\bigoplus_n \left(\begin{tikzcd} H_*(X^{(n-1)}) \arrow[r, "i"] & H_*(X^{(n)}) \arrow[d, "j"] \\ & H_*(X^{(n)}, X^{(n-1)}) \arrow[ul, "\delta"] \end{tikzcd} \right)$$

THIS GIVES RISE TO A SS.

IT IS CALLED THE AHSS WHEN WE USE AN EXTRAORDINARY HOMOLOGY THEORY.

IN OUR CASE MANY GROUPS ARE 0, SS IS DEGENERATE, AND GIVES RISE TO CELLULAR HOMOLOGY.

SINCE $H_m(X^{(n)}, X^{(n-1)}) = 0$ $m \neq n$,

AND THE IMAGE OF A SINGULAR n-CHAIN IS COMPACT,

$$0 \xrightarrow{\cong} H_n(X^{(0)}) \xrightarrow{\cong} H_n(X^{(1)}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_n(X^{(n-1)}) \longrightarrow H_n(X^{(n)}) \longrightarrow H_n(X^{(n+1)}) \xrightarrow{\cong} H_n(X^{(n+2)}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_n(X)$$

$$\begin{tikzcd}
H_{n-1}(X^{(n-1)}) \arrow[d, hook] & H_n(X^{(n)}) \arrow[r, two heads, "i"] \arrow[d, hook, "j"'] & H_n(X^{(n+1)}) \arrow[r, "\cong"] & H_n(X) \\
H_{n-1}(X^{(n-1)}, X^{(n-2)}) & H_n(X^{(n)}, X^{(n-1)}) \arrow[l, "d"] \arrow[ul, "\delta"] & H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[l, "d"] \arrow[ul, "\delta"] &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\delta \arrow[r] \arrow[d, "j", "\cong"'] & H_n(X^{(n)}) \arrow[r, "i"] \arrow[d, "j", "\cong"'] & H_n(X^{(n+1)}) \arrow[r] \arrow[d, dashed, "\cong"] & 0 \\
0 \arrow[r] & \operatorname{im} d \arrow[r] & \ker\delta = \ker d \arrow[r] & \dfrac{\ker d}{\operatorname{im} d} \arrow[r] & 0
\end{tikzcd}$$

FOR $n > 0$,

$$\begin{tikzcd}[column sep=small]
 & H_n(S^n_\alpha) \arrow[rr, "\varphi_\alpha"] & & H_n(X^{(n)}) \arrow[rr] \arrow[dr, "j"] & & H_n\left(X^{(n)}/X^{(n-1)}\right) \arrow[rr] \arrow[dr, "\cong"] & & H_n(S^n_\beta) \arrow[dr, "\cong"] & \\
H_{n+1}(D^{n+1}_\alpha, S^n_\alpha) \arrow[ur, "\partial", "\cong"'] \arrow[rr, "\mathrm{INC}", "I_\alpha"'] & & H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[ur, "\partial"] \arrow[rr, "d"] & & H_n(X^{(n)}, X^{(n-1)}) \arrow[rr, "\cong"] & & H_n\left(X^{(n)}/X^{(n-1)}, *\right) \arrow[rr, "\mathrm{PROJ.}"] & & H_n(S^n_\beta, *)
\end{tikzcd}$$

HOW TO CALCULATE DEGREE?

A SPECIAL CASE:

SUPPOSE $\exists\, x_1, \dots, x_n, y \in S^n$, $U_1, \dots, U_n, V \in S^n$ SUCH THAT

$$f : S^n \longrightarrow S^n$$

$f^{-1}(y) = \{x_1, \dots, x_n\}$

$y \in V$

$x_i \in U_i$, U_i DISJOINT

$$U_i \xrightarrow{\cong} V \cong \mathbb{R}^n$$

$$S^n - \bigcup_i U_i \longrightarrow S^n - V.$$

THEN

$$\begin{tikzcd}
 & S^n \arrow[r, "f"] \arrow[dl, "\cong"'] \arrow[d] & S^n \arrow[d, "\cong"] \\
S^n / S^n - V_i \arrow[d, phantom, "\cong" description] & S^n / S^n - \bigcup_i U_i \arrow[l] \arrow[r] \arrow[d, phantom, "\cong" description] & S^n / S^n - V \arrow[d, phantom, "\cong" description] \\
S^n & \bigvee_i S^n \arrow[l] \arrow[r, "\bigvee_i f|_{x_i}"] & S^n
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z} \arrow[r, "\deg f"] \arrow[dl, "\cong"'] \arrow[d, "{(1,1,\dots,1)}"] & \mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z} & \bigoplus_i \mathbb{Z} \arrow[l, "\pi_i"'] \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

$$\deg f = \sum_i \deg f|_{x_i} \qquad \text{EACH } \deg f|_{x_i} = \pm 1.$$

IN A BIT MORE GENERALITY:

SUPPOSE $\exists x_1, \dots, x_n, y \in S^n$: $f^{-1}(y) = \{x_1, \dots, x_n\}$.

GIVEN OPEN $V \ni y$ (CHOOSE AS SMALL AS CONVENIENT)

CAN FIND DISJOINT $U_1, \dots, U_n$ (AS SMALL AS YOU LIKE)

SUCH THAT $x_i \in U_i$, $f(U_i) \subset V$.

THEN $f(U_i - x_i) \subset V - y$

$$\begin{tikzcd}
 & H_n(S^n) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d] & H_n(S^n) \arrow[d, "\cong"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "\pi_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
H_n(U_j, U_j - x_j) \arrow[u, "\delta_{ij}"'] \arrow[r] & H_n\left(\bigcup_i U_i, \bigcup_i U_i - x_i\right) \arrow[u, "\cong"'] \arrow[r, "f_*"] & H_n(V, V - y) \arrow[u, "\cong"']
\end{tikzcd}$$