ALGEBRAIC TOPOLOGY

- study of topological spaces by associating algebraic invariants (up to homotopy)

  - set : path components π_0

    proof that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic :

    if $f : \mathbb{R} \to \mathbb{R}^2$ was a homeomorphism would induce homeomorphism $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}^2 \setminus \{f(0)\}$

    $\times$ $\mathbb{R} \setminus \{0\}$ not path connected
    $\mathbb{R}^2 \setminus \{f(0)\}$ is.

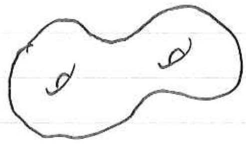
    (exercise : same for S^1 and S^2)

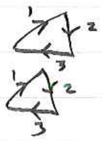
  - number : Euler characteristic

    $\chi(\text{graph}) = \#\text{vertices} - \#\text{edges}$

    $\chi(\text{tree}) = 1 \qquad \chi(n\text{-cycle graph}) = 1 - n$

    determine the homeomorphism type of an orientable (= 2-sided), compact surface

     $= S^2$  $= T$  $= F_2$

       triangulate

    $\chi(S^2) = V - E + F = 3 - 3 + 2 = 2$

    $\chi(T) = 1 - 3 + 2 = 0$

    $\chi(F_2) = 1 - 9 + 6 = -2$

    ...

    $2 - g$

  - groups : fundamental group π_1

    $$\int_\gamma \frac{1}{z}\, dz = 2\pi i\, w(\gamma)$$

    ← winding number of γ

    $\in \mathbb{Z} \cong \pi_1(\mathbb{C} \setminus 0)$

    ↑ generator $t \mapsto e^{2\pi i t}$.

- This course:

homology $\quad H_* = \bigoplus_{n \geq 0} H_n \quad$ graded group, abelian

cohomology $\quad H^* = \bigoplus_{n \geq 0} H^n \quad$ graded ring

why:
- refinement of the Euler characteristic

$$\chi = \sum (-1)^n \underbrace{\operatorname{rank} H_n}_{\substack{\text{number of copies} \\ \text{of } \mathbb{Z}\text{, ignore torsion}}}$$

- gives information on higher dimensional topology

Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ $(n \neq m)$
S^n " " " " S^m $(n \neq m)$

- $H_1 = \dfrac{\pi_1}{[\pi_1, \pi_1]}$ abelianisation of π_1.

$[\pi_1, \pi_1]$: normal subgroup generated by commutators.

- H_n is relatively easy to compute.

I CHAIN COMPLEXES

Let $C_0, C_1, C_2, \ldots$ be abelian groups (or R-modules) and let $\partial_n : C_n \to C_{n-1}$ be group homomorphisms (or R-module homomorphisms)

Assume that $\partial_n \circ \partial_{n+1} = 0$

Then $\quad \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \cdots C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

is a chain complex. The elements of degree n.
We call the elements of

C_n	n-chains	chains of degree n
$Z_n = \ker \partial_n$	n-cycles	cycles ...
$B_n = \operatorname{im} \partial_{n+1}$	n-boundaries	boundaries ...

As $\partial_n \circ \partial_{n+1} = 0$, we have $B_n \subset Z_n \subset C_n$.

So define $H_n(C.,\partial.) = \dfrac{Z_n}{B_n}$

This is the n^{th} homology group of the chain complex $(C.,\partial.)$

A map of chain complexes $(C.,\partial.) \xrightarrow{f} (\tilde{C}.,\tilde{\partial}.)$ is a collection of group (or R-module) homomorphisms $f_n : C_n \to \tilde{C}_n$ s.t.

$\tilde{\partial}_n \circ f_n = f_{n-1} \circ \partial_n$.

ie

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & \cdots \arrow[r] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & 0 \\
\cdots \arrow[r] & \tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"] & \tilde{C}_n \arrow[r, "\tilde{\partial}_n"] & \cdots \arrow[r] & \tilde{C}_1 \arrow[r, "\tilde{\partial}_1"] & \tilde{C}_0 \arrow[r, "\tilde{\partial}_0"] & 0
\end{tikzcd}$$

commutes.

Lemma: A map $f : (C.,\partial.) \to (\tilde{C}.,\tilde{\partial}.)$ of chain complexes induces maps in homology

$$f_* : H_n(C.,\partial.) \to H_n(\tilde{C}.,\tilde{\partial}.)$$
for all n.

Proof: 1) $f_n(Z_n(C.)) \subseteq Z_n(\tilde{C}.)$:

Let $z \in Z_n(C.)$

Then $\tilde{\partial}_n f_n(z) = f_{n-1}\partial_n(z)$ as f is a map of chain complexes

$= f_{n-1}(0)$ as $z \in Z_n(C.)$

$= 0$

2) $f_n(B_n(C.)) \subseteq B_n(\tilde{C}.)$

Let $b \in B_n(C.)$.

Then there exists $c \in C_{n+1}$ with $\partial_{n+1} c = b$.

Hence $f_n(b) = f_n \partial_{n+1}(c) = \tilde{\partial}_{n+1} f_{n+1}(c) = \tilde{\partial}_{n+1}(f_{n+1}(c)) \in B_n(\tilde{C}.)$

3) So induces $f_n : \dfrac{Z_n(C.)}{B_n(C.)} \to \dfrac{Z^n(\tilde{C}.)}{B_n(\tilde{C}.)}$ for all n.

Ex. $C_0 = \mathbb{Z} = \langle v \rangle \qquad \partial_0 = 0$

$C_1 = \mathbb{Z} = \langle a \rangle \qquad \partial_1 = 0$

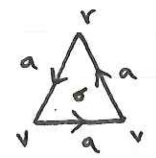
$C_2 = \mathbb{Z} = \langle \sigma \rangle \qquad \partial_2(\sigma) = 3a$

(conventionally $C_n = 0 \quad n > 2$ since these are not given)

$$H_0 = \frac{Z_0}{B_0} = \frac{\mathbb{Z}}{0} = \mathbb{Z}$$

$$H_1 = \frac{Z_1}{B_1} = \frac{\mathbb{Z}}{3\mathbb{Z}} = \mathbb{Z}_3$$

$$H_2 = \frac{Z_2}{B_2} = \frac{0}{0} = 0$$



Ex. With coefficients in $\mathbb{Q}$:

$C_0 = C_1 = C_2 = \mathbb{Q}$

$H_0 = \mathbb{Q} \qquad H_1 = 0 \qquad H_2 = 0$

(since $3\mathbb{Q} = \mathbb{Q}$)

Ex. With coefficients in $\frac{\mathbb{Z}}{3\mathbb{Z}} = \mathbb{F}_3$

$C_0 = C_1 = C_2 = \mathbb{F}_3$

$H_0 = \mathbb{F}_3 \qquad H_1 = \mathbb{F}_3 \qquad H_2 = \mathbb{F}_3$

$(3\mathbb{F}_3 = 0)$

2 Δ-complexes

Let $\{v_0, \dots, v_n\} \in \mathbb{R}^{n+k}$ be s.t.

$v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ linearly independent

The oriented n-simplex $[v_0, v_1, \dots, v_n]$ is the convex hull (smallest convex set containing) of $\{v_0, \dots, v_n\}$, together with an orientation, i.e. sign determined by the ordering of $\{v_0, \dots, v_n\}$. So

$$[v_0, v_1, v_2] = -[v_0, v_2, v_1]$$
$$= [v_2, v_0, v_1]$$

?? Just important to remember the vertices have an ordering

Ex. The standard, Δ^n, n-simplex is given by the standard basis in $\mathbb{R}^{n+1}$.

$$\Delta^n = \{ \sum t_i e_i : \sum t_i = 1, \; t_i \geq 0 \}$$

Δ^0 

There are canonical homeomorphisms

$$\sigma : \Delta^n \longrightarrow [v_0, \dots, v_n]$$
$$\sum t_i e_i \longmapsto \sum t_i v_i .$$

A face of an n-simplex $[v_0, \dots, v_n]$ is any simplex determined by a (non-empty) subset of $\{v_0, \dots, v_n\}$ with the induced orientation.

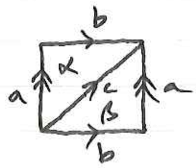
induced ordering...

A Δ-complex X, is the quotient space of a disjoint union of simplices under the identification of some faces via the canonical map.

(with the given orientation)

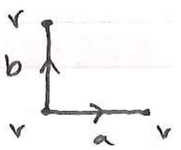
the canonical map preserves ordering.

Ex. Torus T



SEE HATCHER for clarity on all this. Characteristic maps.

$S^1 \vee S^1$



Remark: Any realisation of a simplicial set is a Δ-complex.

3 Simplicial Homology.

Let $X_\cdot$ be a Δ-complex.

$$\Delta_n(X_\cdot) := \text{free abelian group on all } n\text{-simplices of } X_\cdot \ (X_n)$$
$$= \left\{ \sum_{\alpha \in X_n} n_\alpha \alpha \ : \ n_\alpha \in \mathbb{Z},\ \text{only finitely many } n_\alpha \text{ are non-zero} \right\}$$

$$\partial_n : \Delta_n(X_\cdot) \longrightarrow \Delta_{n-1}(X_\cdot)$$
$$\alpha = [v_0, v_1, \dots, v_n] \longmapsto \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

Key Lemma: $\partial_{n-1} \circ \partial_n = 0$

"boundary of a boundary is empty"

Proof: It is enough to check this on generators:

$$\partial \circ \partial [v_0, \dots, v_n] = \partial \left(\sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n] \right)$$

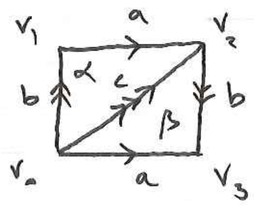
omit subscript

$$= \sum_{i=0}^{n} \left[\sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] + \sum_{j>i} (-1)^{i+j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n] \right]$$

$$= 0.$$

Definition: Simplicial homology $H_n^\Delta(X_\cdot) := H_n(\Delta_\cdot(X_\cdot), \partial_\cdot)$

Ex: Klein bottle.



$v_0 = v_1 = v_2 = v_3 = v$

$$\Delta_0 = \langle v \rangle = \mathbb{Z}$$

$$\Delta_1 = \langle a, b, c \rangle = \mathbb{Z}^3$$

$$\Delta_2 = \langle \alpha, \beta \rangle = \mathbb{Z}^2$$

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

$$\partial_1(a) = \partial_1([v_0, v_3]) = [v_3] - [v_0] = 0$$
$$\partial_1(b) = 0$$
$$\partial_1(c) = 0$$

$$\partial_2(\alpha) = \partial_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$
$$= a - c + b$$

$$\partial_2(\beta) = c - a + b$$

$$H_0 = \frac{Z_0}{B_0} = \frac{\mathbb{Z}}{0} = \mathbb{Z}$$

$$H_1 = \frac{Z_1}{B_1} = \frac{\mathbb{Z}^3}{\langle a-c+b, c-a+b \rangle}$$

$$H_2 = \frac{Z_2}{B_2} = \frac{0}{0} = 0$$ (since $a-c+b$, $c-a+b$ are linearly independent)

$$\partial_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \underset{}{\overset{C_2 \mapsto C_2 + C_1}{\rightsquigarrow}} \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$\underset{\substack{R_2 \mapsto R_2 - R_1 \\ R_3 \mapsto R_1 + R_3}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$H_1 = \frac{\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle}{\langle \tilde{a}, 2\tilde{b} \rangle} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}$$

new basis $\tilde{\alpha} = \alpha$ $\tilde{\beta} = \alpha + \beta$

$$\partial(\tilde{\alpha}) = a + b - c = \tilde{a}$$
$$\partial(\tilde{\beta}) = 2b = 2\tilde{b}$$

II Singular Homology

Immediate goal :
- continuous maps induce maps on H_*
- H_* is an invariant of homotopy

Let X be any topological space. A continuous map $\sigma : \Delta^n \to X$ is called a singular n-simplex.

$C_n(X) :=$ free abelian group on singular n-simplices
$=$ singular n-chains
$= \left\{ \sum n_\alpha \sigma_\alpha : n_\alpha \in \mathbb{Z},\ \sigma_\alpha : \Delta^n \to X, \text{ finitely many } n_\alpha \text{ are non-zero} \right\}$

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$
$$\sigma_\alpha \longmapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$
$$= \sigma_\alpha|_{\partial[v_0, \dots, v_n]}$$
$$\left(\Delta^n = [v_0, \dots, v_n] \right)$$

N.B. $\partial_{n-1} \circ \partial_n = 0$

Definition : Singular homology $H_n(X) = H_n(C_\cdot(X), \partial_\cdot)$

$$\Delta_n(X_\cdot) \hookrightarrow C_n(X_\cdot)$$

simplex $\longmapsto$ its characteristic map

Δ-set $\alpha \longmapsto \sigma_\alpha : \Delta^n \to X$ the canonical imbedding.

This is an injection of chain complexes.

Theorem : The induced map in homology is an isomorphism

$$H_n(\Delta_\cdot(X_\cdot), \partial_\cdot) \xrightarrow{\cong} H_n(C_\cdot(X_\cdot), \partial_\cdot)$$

Proof : postpone ...

Ex. $X = pt.$

There is only one map $c^n : \Delta^n \to X$, the constant map.

$$\cdots \to \underset{c^2}{\mathbb{Z}} \to \underset{c^1}{\mathbb{Z}} \to \underset{c^0}{\mathbb{Z}} \to 0$$

$$\partial_n(c^n) = \sum_{i=0}^{n} (-1)^i \, c^n\big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$= \begin{cases} 0 & n \text{ is odd} \\ c^{n-1} & n \text{ is even} \end{cases} \qquad (n>0)$$

$$\xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

Recap: Simplicial $\longleftrightarrow$ Singular

- easy computation
- functoriality
- homotopy invariance

Functoriality

space $X \longmapsto C_\cdot(X) \longmapsto H_*(X) = \bigoplus_{n \geq 0} H_n(X)$

singular chain complex

$f: X \to Y \longmapsto f_\# : C_\cdot(X) \to C_\cdot(Y) \longmapsto f_* : H_n(X) \to H_n(Y)$

continuous map

$(\sigma : \Delta^n \to X)$
$\longmapsto (f \circ \sigma : \Delta^n \to Y)$

$$f_\#(\partial\sigma) = \partial(f_\# \sigma)$$

Furthermore, if $g: Y \to Z$ then $(g \circ f)_\# = g_\# \circ f_\#$

and so $(g \circ f)_* = g_* \circ f_*$

and $(id : X \to X)_\# = id_{C.(X)} \Rightarrow (id : X \to X)_* = id_{H_n(X)}$

We say, homology is a functor from topological spaces to graded abelian groups.

5 Homotopy Invariance

5a (Algebraic homotopy invariance)

Definition: Two chain maps $\phi, \psi : (C., \partial.) \to (\tilde{C}., \tilde{\partial}.)$ are chain homotopic if there exists a group homomorphism $h_n : C_n \to \tilde{C}_{n+1}$ s.t.

$$\tilde{\partial}_{n+1} \circ h_n + h_{n-1} \circ \partial_n = \phi_n - \psi_n$$

We call h a chain homotopy.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"] & C_{n-1} \arrow[dl, "h_{n-1}"] \\
\tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r, "\tilde{\partial}_n"'] & \tilde{C}_{n-1}
\end{tikzcd}$$

Proposition: If ϕ, ψ are chain homotopic then

$$\phi_* = \psi_* : H_n(C., \partial.) \to H_n(\tilde{C}., \tilde{\partial}.)$$

Proof: Let $z \in Z_n = \ker \partial_n$

Then
$$\begin{aligned}
\phi_n(z) - \psi_n(z) &= (\tilde{\partial}_{n+1} \circ h_n + h_{n-1} \circ \partial_n)(z) \\
&= \tilde{\partial}_{n+1}(h_n(z)) + h_{n-1}(\underbrace{\partial_n(z)}_{=0}) \\
&\in \operatorname{im} \tilde{\partial}_{n+1}
\end{aligned}$$

So $\phi_n(z) - \psi_n(z) = 0$ in $H_n(\tilde{C}., \tilde{\partial}.) = \dfrac{\tilde{Z}_n}{\tilde{B}_n}$.

Main Example: Let $i_0, i_1 : X \longrightarrow X \times [0,1]$ be the inclusions

$x \longmapsto (x,0)$ and $x \longmapsto (x,1)$

Then there exists a chain homotopy h between

$$(i_0)_\#, (i_1)_\# : C_\cdot(X) \longrightarrow C_\cdot(X \times [0,1])$$

We can construct one as follows...

Let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$
$\Delta^n \times \{1\} = [w_0, \dots, w_n]$.

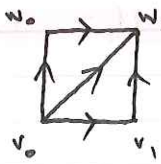
Divide $\Delta^n \times [0,1]$ into $n+1$, $(n+1)$-simplices of the form

$$s_i = [v_0, \dots, v_i, w_i, \dots, w_n].$$

Put $\Gamma = \sum_{i=0}^{n} (-1)^i s_i \in C_{n+1}(\Delta^n \times [0,1])$

$\partial \Gamma =\sim$ faces of $\partial \Delta^n \times [0,1] \cup \Delta^n \times \partial [0,1]$.

e.g.



$$\Gamma = [v_0, w_0, w_1] - [v_0, v_1, w_1]$$

$$\partial \Gamma = [w_0, w_1] + [v_0, w_0] - [v_1, w_1] - [v_1, w_1]$$

$=\sim$ 

Let $\sigma : \Delta^n \to X \in C_n(X)$.

Then $\sigma \times 1 : \Delta^n \times [0,1] \longrightarrow X \times [0,1]$.

Define $h_n : C_n(X) \longrightarrow C_{n+1}(X \times [0,1])$

$$\sigma \longmapsto (\sigma \times 1)_\#(\Gamma) = \sum_{i=0}^{n} (-1)^i (\sigma \times 1)\big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}.$$

Then $\partial_{n+1} h_n(\sigma) + h_{n-1} \partial_n (\sigma)$

$$= \sum_{i=0}^{n} \Big[\sum_{j \leq i} (-1)^{i+j} (\sigma \times 1)\big|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$
$$+ \sum_{j \geq i} (-1)^{i+j+1} (\sigma \times 1)\big|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]} \Big]$$

$$+ \sum_{j=0}^{n} \Big[\sum_{i<j} (-1)^{i+j} \, \sigma|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

$$+ \sum_{i>j} (-1)^{i+j+1} \, \sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]} \Big]$$

$$= (\sigma \times 1)|_{[w_0,\dots,w_n]} - (\sigma \times 1)|_{[v_0,\dots,v_n]} \qquad \text{(after some cancellation)}$$

$$= (i_1)_\#(\sigma) - (i_0)_\#(\sigma)$$

$\Rightarrow \quad (i_1)_\# - (i_0)_\# = \partial_{n+1} h_n + h_{n-1} \partial_n \quad \Rightarrow \quad h$ is a chain homotopy.

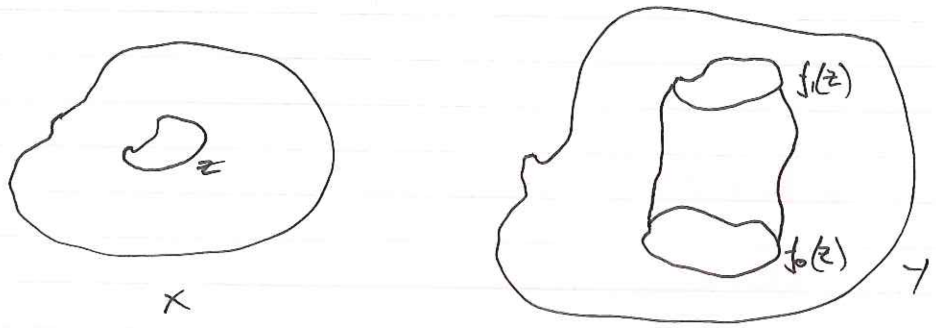
(Topological homotopy invariance)

Definition: Two maps $f_0, f_1 : X \to Y$ are homotopic ($f_0 \sim f_1$) if there exists $F : X \times [0,1] \to Y$ s.t. $F|_{X \times \{0\}} = f_0$, $F|_{X \times \{1\}} = f_1$. F is called a homotopy.

Definition: Two spaces X and Y are called homotopic ($X \simeq Y$) if there exists $f : X \to Y$, $g : Y \to X$ s.t. $gf \simeq id_X$, $fg \simeq id_Y$.

Theorem: If $f_0 \sim f_1 : X \to Y$, then $(f_0)_* = (f_1)_* : H_*(X) \to H_*(Y)$

Geometric intuition



$f_1(z) - f_0(z)$ is a boundary.

Corollary: If $X \simeq Y$ then $H_*(X) = H_*(Y)$.

Ex. A contractible space X (ie $X \sim pt$) has homology

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Theorem: if $f \simeq g$ then $f_* = g_* : H_n(X) \to H_n(Y)$

Proof: Let F be a homotopy.

Then $f = F \circ i_0$

and $g = F \circ i_1$

$\Rightarrow f_* = F_* \circ i_{0*}$

and $g_* = F_* \circ i_{1*}$

But $i_{0*} = i_{1*}$ $\Rightarrow$ $f_* = g_*$

Corollary: $X \simeq Y \Rightarrow H_n(X) \cong H_n(Y)$ for all n.

Proof: Let $f: X \to Y$, $g: Y \to X$ s.t.

$id_X \simeq g \circ f$ $\quad id_Y \simeq f \circ g$

Then $(id_X)_* = g_* \circ f_*$ $\quad (id_Y)_* = f_* \circ g_*$

$\Rightarrow f_* : H_n(X) \longleftrightarrow H_n(Y) : g_*$

are inverses to one another.

6 Snake Lemma + relative homology.

6a (Algebraic relative homology)

A sequence of (abelian) groups (or R-modules...)

$$A_n \xrightarrow{\partial_n} \dots \to A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

is called exact, if at every point we have $\operatorname{im} \partial_{n+1} = \ker \partial_n$
(is a chain complex, and $H_n = 0$ for all n)

An exact sequence $0 \to A \to B \to C \to 0$ is called a short exact sequence.

This says $A \to B$ is an injection
$B \to C$ is a surjection

and $C \cong \frac{B}{A}$ when we are identifying A and its image under $A \to B$.

A sequence $0 \to A_\cdot \xrightarrow{i} B_\cdot \xrightarrow{j} C_\cdot \to 0$ of chain complexes is called exact if each

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$$

is short exact.

Theorem: There exist connecting homomorphisms

$$\delta : H_n(C_\cdot) \to H_{n-1}(A_\cdot) \quad \text{s.t.}$$

the following is a long exact sequence:

$$\to H_n(A_\cdot) \xrightarrow{i} H_n(B_\cdot) \xrightarrow{j} H_n(C_\cdot) \xrightarrow{\delta} H_{n-1}(A_\cdot) \xrightarrow{i} \dots$$

$$\dots \xrightarrow{\delta} H_0(A_\cdot) \xrightarrow{i} H_0(B_\cdot) \xrightarrow{j} H_0(C_\cdot) \to 0$$

Proof:

$$\begin{tikzcd}
A_{n+1} \arrow[r, "\partial"] \arrow[d, hook, "i"'] & A_n \arrow[r] \arrow[d, hook] & A_{n-1} \arrow[d, hook] \\
B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"'] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[d] \\
C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r] & C_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \arrow[d, mapsto] \\
b \arrow[r, mapsto] \arrow[d, mapsto] & \partial b \arrow[d, mapsto] \\
c \arrow[r, mapsto] & 0
\end{tikzcd}$$

Define δ: Let $[c] \in H_n(C_\cdot)$. Then $\partial c = 0$ and there exist $b \in B_n$ s.t. $j(b) = c$. So $j(\partial b) = \partial(jb) = \partial c = 0$ and hence there exists $a \in A_{n-1}$ s.t. $i(a) = \partial b$.
Define $\delta[c] = [a]$

Well-defined: a is a cycle.
$i(\partial a) = \partial(ia) = \partial(\partial b) = 0$
$\Rightarrow \partial a = 0$ as i is injective

choice of c: Suppose $[c'] = [c]$.

Then there exists $c'' \in C_{n+1}$ s.t. $\partial c'' = c - c'$, and there exists $b'' \in B_{n+1}$ s.t. $j(b'') = c''$.

Now $j(\partial b'') = \partial(j b'') = \partial c'' = c - c' = j(b) - c'$

$\Rightarrow$ $c' = j(b - \partial b'')$.

logically better

So changing c to c' changes b to $b' = b - \partial b''$ and then $\partial b' = \partial b - \partial\partial b'' = \partial b$, so that ∂b and hence a is unchanged.

choice of lift of c in b:

Suppose $j(b') = c$. Then $b - b' \in \ker j$
$\Rightarrow$ there exists $a'' \in A_n$ s.t. $i(a'') = b - b'$.

Now $i(\partial a'') = \partial(i a'') = \partial b - \partial b'$
$\Rightarrow$ $\partial b' = \partial b - i(\partial a'') = i(a - \partial a'')$

So changing b to b' changes a to $a' = a - \partial a''$ which leaves $[a]$ unchanged.

Exactness: at $H_n(C)$: $\operatorname{im} j \subseteq \ker \delta$

Let $j[b] = [j(b)] \in \operatorname{im} j$.

Then $\delta[j(b)] = [a]$ where $i(a) = \partial b$.

But $[b] \in H_n(B) \Rightarrow \partial b = 0 \Rightarrow [a] = 0$

$\operatorname{im} j \supseteq \ker \delta$

Let $[c] \in \ker \delta$, and choose $b \in B_n$ and $a \in A_{n-1}$ s.t. $j(b) = c$ and $i(a) = \partial b$.
Then $[a] = \delta[c] = 0 \Rightarrow a = \partial a'$ for some $a' \in A_n$.

Now $\partial(b - i(a')) = \partial b - i(\partial a') = \partial b - i(a) = 0$
$\Rightarrow$ $[b - i(a')] \in H_n(B)$ and $j[b - i(a')] = [j(b)]$

$= [c]$.

Exactness at $H_n(B)$: $ji = 0$ before passing to homology, so $ji = 0$ on homology

$\Rightarrow$ $\operatorname{im} i \subseteq \ker j$.

If $j[b] = 0$, then $j(b) = \partial c'$ for some $c' \in C_{n+1}$. There exists $b' \in B_{n+1}$ s.t. $j(b') = c'$ and now

$$j(\partial b') = \partial(jb') = \partial c' = j(b).$$

So $b - \partial b' \in \ker j$ $\Rightarrow$ there exists $a \in A_n$ s.t. $i(a) = b - \partial b'$.

and $i\partial(a) = \partial i(a) = \partial b - \partial\partial b' = 0$

$\Rightarrow \partial a = 0 \Rightarrow [a] \in H_n(A)$.

$i[a] = [b - \partial b'] = [b]$ $\Rightarrow$ $\ker j \subseteq \operatorname{im} i$.

Exactness at $H_n(A)$: Let $[c] \in H_{n+1}(C)$. Choose $b \in B_{n+1}$ and $a \in A_n$ s.t. $j(b) = c$ and $i(a) = \partial b$.

Then $i\delta[c] = i[a] = [\partial b] = 0$ in $H_n(B)$

$\Rightarrow$ $\operatorname{im} \delta \subseteq \ker i$.

Let $[a] \in \ker i$. Then $i(a) = \partial b$ for some $b \in B_{n+1}$. Let $c = j(b) \in C_{n+1}$.

Then $\partial c = \partial(jb) = j(\partial b) = j(ia) = 0$

$\Rightarrow$ $[c] \in H_{n+1}(C)$ and $\delta[c] = [a]$.

$\Rightarrow$ $\ker i \subseteq \operatorname{im} \delta$.

ASIDE :

Recap chain complex $C.$: $\to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to$ $\partial\partial = 0$

map between chain complexes

$f : C. \to C.'$:
$$\begin{tikzcd} \to C_{n+1} \arrow[r, "\partial"] \arrow[d, "f"] & C_n \arrow[r, "\partial"] \arrow[d, "f"] & C_{n-1} \to \arrow[d, "f"] \\ \to C'_{n+1} \arrow[r, "\partial"] & C'_n \arrow[r, "\partial"] & C'_{n-1} \to \end{tikzcd}$$

commuting

short exact sequence of chain complexes

$0 \to A. \to B. \to C. \to 0$: shorthand for

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

rows exact, columns chain complexes, maps commute

A map between short exact sequence of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & A. \arrow[r, "i"] \arrow[d, "\alpha"] & B. \arrow[r, "j"] \arrow[d, "\beta"] & C. \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A.' \arrow[r, "i"'] & B.' \arrow[r, "j"'] & C.' \arrow[r] & 0 \end{tikzcd}$$

all maps are maps of chain complexes and they commute.

In particular
$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A'_n \arrow[r] & B'_n \arrow[r] & C'_n \arrow[r] & 0 \end{tikzcd}$$
commutes for each n.

So we have
$$\begin{tikzcd} H_n(A.) \arrow[r, "i"] \arrow[d, "\alpha"] & H_n(B.) \arrow[r, "j"] \arrow[d, "\beta"] & H_n(C.) \arrow[r, "\delta"] \arrow[d, "\gamma"] & H_{n-1}(A.) \arrow[d, "\alpha"] \\ H_n(A.') \arrow[r, "i"] & H_n(B.') \arrow[r, "j"] & H_n(C.') \arrow[r, "\delta"] & H_{n-1}(A.') \end{tikzcd}$$

$\beta i = i\alpha$ before passing to homology so holds on homology
$\gamma j = j\beta$ " " " " " " " "

Do we have $\alpha\delta = \delta\gamma$??

P.T.O.

Let $[c] \in H_n(C)$.

Choose $b \in B_n$ s.t. $j(b) = c$ and $a \in A_{n-1}$ s.t. $i(a) = \partial b$.

Then $j(b) = c \Rightarrow \gamma j(b) = \gamma(c) \Rightarrow j\beta(b) = \gamma(c)$

and $i(a) = \partial b \Rightarrow \beta i(a) = \beta\partial(b) \Rightarrow i\alpha(a) = \partial\beta(b)$.

In summary
$$j(b) = c \quad \text{and} \quad i(a) = \partial b$$
$$j(\beta(b)) = \gamma(c) \quad \text{and} \quad i(\alpha(a)) = \partial(\beta(b)).$$

$\Rightarrow \quad \alpha\delta[c] = \alpha[a] = [\alpha(a)]$

and $\quad \delta\gamma[c] = \delta[\gamma(c)] = [\alpha(a)]$

$$\Rightarrow \quad \alpha\delta = \delta\gamma.$$

The maps between the long exact sequences commute.

Exercise 1.4:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "\partial"', "\cong"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 & D. \\
\cdots & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & C_{n-2} \arrow[d] & \cdots & C. \\
\cdots & C_{n+1} \arrow[r] & \frac{C_n}{A} \arrow[r] & \frac{C_{n-1}}{B} \arrow[r] & C_{n-2} & \cdots & \tilde{C}.
\end{tikzcd}$$

short exact sequence of chain complexes

$\rightsquigarrow$ long exact sequence

$$\cdots \to H_n D. \to H_n C. \to H_n \tilde{C}. \to H_{n-1} D. \to \cdots$$

As $H_k D. = 0$ for all $k \geq 0$

CLEVER!!

so $H_k C. \xrightarrow{\cong} H_k \tilde{C}.$

Relative Homology of spaces

Let A be a subspace of X.

Define relative chains

$$C_n(X,A) := \frac{C_n(X)}{C_n(A)}$$

and $\bar{\partial}_n : C_n(X,A) \longrightarrow C_{n-1}(X,A)$

$$\bar{c} \longmapsto \overline{(\partial c)}$$

where $c \in C_n(X)$ and $\bar{c}$ is the class in $\frac{C_n(X)}{C_n(A)}$. This is well-defined as

$$\partial(C_n(A)) \subset C_{n-1}(A)$$

N.B. $\tilde{\partial}_n \circ \tilde{\partial}_{n+1} = 0$ since $\partial_n \circ \partial_{n+1} = 0$

Define relative homology

$$H_n(X,A) := H_n(C.(X,A), \bar{\partial}.)$$

Note: a cycle in $C_n(X,A)$ is represented by a chain $c \in C_n(X)$ s.t. $\partial c \in C_{n-1}(A)$, a relative cycle.

We will see that for "good" (X,A)

$$H_n(X,A) = H_n\left(\frac{X}{A}\right) \quad n > 0.$$

Theorem: There is a long exact sequence

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\delta} H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\delta} \cdots$$

with $\delta[\bar{c}] = [\partial c]$

Ex: $A = pt = * \subset X$

$$H_n(*) \to H_n(X) \to H_n(X,*) \xrightarrow{\delta} H_{n-1}(*) \to \cdots$$
$$\to H_1(X,*) \to H_0(*) \hookrightarrow H_0(X)$$
$$\to H_0(X,*) \to 0$$

for $n > 1$, $H_n(*) = H_{n-1}(*) = 0$

$$\Rightarrow H_n(X,*) \cong H_n(X)$$

Claim: $H_0(*) \hookrightarrow H_0(X)$

Proof: use functoriality:

We have a unique map $j: X \to *$

and then $* \hookrightarrow X \xrightarrow{j} *$

is the identity $\Rightarrow$

$$H_n(*) \to H_n(X) \xrightarrow{j_*} H_n(*)$$

is the identity $\Rightarrow$ $H_n(*) \to H_n(X)$ is

injective.

So $\ker(H_0(*) \to H_0(X)) = 0$

$\Rightarrow$ $\operatorname{im}(\delta : H_1(X,*) \to H_0(*)) = 0$

$\cdots \Rightarrow$ $H_1(X) \cong H_1(X,*)$

look up split exact

Also, $0 \to H_0(*) \hookrightarrow H_0(X) \to H_0(X,*) \to 0$
is "split exact" by the claim.
So $H_0(X) \cong H_0(*) \oplus H_0(X,*) \cong \mathbb{Z} \oplus H_0(X,*)$

and $H_0(X,*) \cong \ker\left(H_0(X) \xrightarrow{g_*} H_0(*)\right)$

Definition: The reduced homology of X is defined by

$$\tilde{H}_n(X) := \ker\left(g_* : H_n(X) \to H_n(*)\right)$$

(By above, $\tilde{H}_n(X) \cong H_n(X,*)$ for $n \geq 0$)

see equiv. definition in Hatcher

Ex: $(X,A) = (D^k, S^{k-1})$

$$H_n(D^k) \to H_n(D^k, S^{k-1}) \xrightarrow{\delta} H_{n-1}(S^{k-1}) \to H_{n-1}(D^k)$$

$$\cdots \to H_1(D^k, S^{k-1}) \to H_0(S^{k-1}) \to H_0(D^k) \to H_0(D^k, S^{k-1}) \to 0$$

So $H_n(D^k, S^{k-1}) \cong H_{n-1}(S^{k-1})$ for $n \geq 1$.

7 Five Lemma + Locality

7a Five Lemma

In the commutative diagram below, if the rows are exact, then γ is an isomorphism

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"', "\cong"] & B \arrow[r] \arrow[d, "\beta"', "\cong"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r, "\partial"] \arrow[d, "\delta"', "\cong"] & E \arrow[d, "\varepsilon"', "\cong"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Proof: (i) β, δ surjective, ε injective $\Rightarrow$ γ surjective

(ii) β, δ injective, α surjective $\Rightarrow$ γ injective.

Proof: i)

$$\begin{tikzcd}
c \arrow[r, mapsto] & d \arrow[r, mapsto] \arrow[d, mapsto] & \partial d = 0 \arrow[d, mapsto] \\
c' \arrow[r, mapsto] & \partial c' \arrow[r, mapsto] & \varepsilon \partial d = \partial\partial c' = 0
\end{tikzcd}$$

Let $c' \in C'$. As δ is surjective there exists $d \in D$ s.t. $\delta d = \partial c'$.

By commutativity

$$\varepsilon(\partial d) = \partial(\delta d) = \partial\partial c' = 0$$

By injectivity of ε, $\partial d = 0$

By exactness there exists $c \in C$ s.t. $\partial c = d$.

So
$$\begin{aligned}\partial(c' - \gamma c) = \partial c' - \partial \gamma c &= \partial c' - \delta \partial c \\ &= \partial c' - \delta d \\ &= \partial c' - \partial c' = 0\end{aligned}$$

$$\begin{tikzcd}
b \arrow[r, mapsto] \arrow[d, mapsto] & \partial b \arrow[d, mapsto] \\
b' \arrow[r, mapsto] & c' - \gamma c
\end{tikzcd}$$

By exactness, and surjectivity of β

there exists $b \in B$, $b' \in B'$ s.t.

$$\partial(\beta b) = \partial(b') = c' - \gamma c$$

By commutativity $\partial(\beta b) = \gamma(\partial b) = c' - \gamma c$

$$\Rightarrow \quad c' = \gamma(\partial b + c).$$

ii) Exercise ...

$$\begin{tikzcd}
a \arrow[r, mapsto, dashed] \arrow[d, mapsto] & b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[r, mapsto] \arrow[d, mapsto] & \partial c = 0 \arrow[d, mapsto] \\
a' \arrow[r, mapsto] & \beta b \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

7a Five Lemma

7b Locality (global information can be assembled from local info.)

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a collection of subsets of X s.t. $X = \bigcup_{i \in I} \mathring{U}_i$

Define $C_\cdot^{\mathcal{U}}(X) \subseteq C_\cdot(X)$ be the subcomplex generated by n-simplices σ s.t. $\sigma(\Delta^n) \subset U_i$ for some $i \in I$

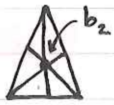
Theorem: the inclusion $C_\cdot^{\mathcal{U}}(X) \hookrightarrow C_\cdot(X)$ induces an isomorphism in homology

$$H_n(C_\cdot^{\mathcal{U}}(X)) \to H_n(C_\cdot(X)) = H_n(X)$$

Proof: (sketch of ideas)

1) Barycentric subdivision

$S(\Delta^1) =$ 

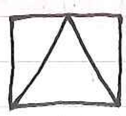
$S(\Delta^2) =$ 

every $(n-1)$-simplex of $S(\partial\Delta^n)$ is joined to b_n.

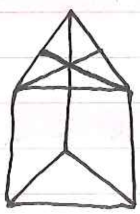
Can check $\partial(S\Delta^n) = S(\partial\Delta^n)$

$\Rightarrow$ chain map $S : C_n(X) \to C_n(X)$
$\sigma \mapsto S\sigma$

2) Chain homotopy T:

$T(\Delta^1 \times [0,1]) =$ 

top: barycentric subdivision
bottom: standard simplex.

$T(\Delta^2 \times [0,1]) =$ 

+ some more lines

every n-simplex of $\Delta^n \times \{0\}$ and $T(\partial\Delta^n \times [0,1])$ is joined to $b_n \in \Delta^n \times \{1\}$.

Can check $\partial T + T\partial = id - S$ when we define chain homotopy $T : C_n(X) \longrightarrow C_{n+1}(X)$

$$\sigma \mapsto T(\Delta^n \times I \xrightarrow{pr} \Delta^n \xrightarrow{\sigma} X)$$

3) "Repeat S" to give $\hat{S} : C.(X) \longrightarrow C^{\mathcal{U}}_.(X)$

do some manipulation trick.

check gives inverse homotopy ...

6 Excision (what distinguishes homotopy groups from homology)

Theorem: Let $Z \subset A \subset X$ with $\bar{Z} \subset \mathring{A}$.
Then
$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A)$$
is an isomorphism for all $n \geq 0$
The map being induced by $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$

Proof: Let V be s.t. $A \subseteq V \subset X$ and $\bar{A} \subseteq \mathring{V}$.

Put $U = X \setminus A$, then $V \cap U = V \setminus A$

Put $\mathcal{U} = \{U, V\}$ check $\mathring{U} \cup \mathring{V} = X$

Then
$$\frac{C_n(U)}{C_n(V \cap U)} \cong \frac{C_n^{\mathcal{U}}(X)}{C_n(V)}$$
induced by inclusion

We have a map of long exact sequences $\frac{C^{\mathcal{U}}_.(X)}{C.(V)}$

$$\begin{array}{ccccccccc}
H_k(V) & \to & H_k(C^{\mathcal{U}}_.(X)) & \to & H_k\left(\frac{C^{\mathcal{U}}_.(X)}{C.(V)}\right) & \to & H_{k-1}(V) & \to & H_{k-1}(C^{\mathcal{U}}_.(X)) \\
\| & & \downarrow \cong & & \downarrow & & \| & & \downarrow \cong \\
H_k(V) & \to & H_k(X) & \to & H_k(X, V) & \to & H_{k-1}(V) & \to & H_{k-1}(X)
\end{array}$$

By five-lemma $H_k(U, V \cap U) \cong H_k\left(\frac{C^{\mathcal{U}}(X)}{C.(V)}\right) \xrightarrow{\cong} H_k(X, V)$

$\qquad \| $

$H_k(X \setminus A, V \setminus A)$

Let $A \subset X$ be a subspace. A map $r : X \to A$ is a retraction if $r|_A = \mathrm{id}_A$.

If r is a homotopy equivalence (rel A), then r is a deformation retraction.

A pair (X, A) is good if A is non-empty, closed subset of X s.t. A is a deformation retraction of a neighbourhood V (i.e. V is open in X and contains A) in X.

Ex 1 : $X = S^1 \vee S^1$

$A = S^1$

$V =$

Ex 2 : $X = D^n$

$A = \partial D^n$

$V = \partial D^n$ thickened.

Define the quotient space $\frac{X}{A}$ as the quotient of X under the relation $x \sim y$ iff $(x, y \in A)$ or $(x = y)$.

Define $q : X \to \frac{X}{A}$ to be the quotient map.

Corollary : For (X, A) good, the quotient map

$$q : (X, A) \to \left(\frac{X}{A}, \frac{A}{A}\right)$$

induces an isomorphism

$$H_n(X, A) \xrightarrow{\cong} H_n\left(\frac{X}{A}, \frac{A}{A}\right) \cong \tilde{H}_n\left(\frac{X}{A}\right).$$

Proof : there exists $V \supset A$ s.t. A is a deformation retract of V, V open. Consider the commutative diagram

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "i"] \arrow[d, "q"] & H_n(X, V) \arrow[d] & H_n(X \setminus A, V \setminus A) \arrow[l, "\cong"', "\text{excision}"] \arrow[d, "="] \\
H_n\left(\frac{X}{A}, \frac{A}{A}\right) \arrow[r, "i"] & H_n\left(\frac{X}{A}, \frac{V}{A}\right) & H_n\left(\frac{X}{A} \setminus \frac{A}{A}, \frac{V}{A} \setminus \frac{A}{A}\right) \arrow[l, "\cong"', "\text{excision}"]
\end{tikzcd}$$

Deformation retraction of V onto A induces deformation retraction of $\frac{V}{A}$ onto $\frac{A}{A}$

Top left is $\cong$ by q's, sheet 3, sim. bottom left

$A \overset{\cong}{\hookrightarrow} V$, $\frac{A}{A} \overset{\cong}{\hookrightarrow} \frac{V}{A}$.

Hence, natural map is middle and map on left is an isomorphism.

Ex: Claim $\tilde{H}_n(S^k) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$

Proof: $(D^k, \partial D^k)$ is good

By induction: for $k = 0$ this is true

$H_n(S^0) = H_n(2 \text{ pts}) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 0 \end{cases}$

$\tilde{H}_n$ knocks off a $\mathbb{Z}$ in dimension 0.

Consider L.E.S. $k \geq 1$

$$0 = \tilde{H}_n(D^k) \to H_n(D^k, \partial D^k) \to \tilde{H}_{n-1}(\partial D^k) \to \dots$$

$$0 = \tilde{H}_1(D^k) \to H_1(D^k, \partial D^k) \to \tilde{H}_0(\partial D^k) \to \tilde{H}_0(D^k) = 0$$

$$\to H_0(D^k, \partial D^k) \to 0.$$

$\Big(k = 1 \quad \partial D^1 = 2 \text{ pts} \Rightarrow \tilde{H}_0(\partial D^1) \cong \mathbb{Z}$

$\Rightarrow H_1(D^1, \partial D^1) \cong \mathbb{Z} \cong \tilde{H}_1(S^1)\Big)$

Also for all $k \geq 1$

$$H_0(D^k, \partial D^k) \cong \tilde{H}_0(S^k) = 0$$

(Also $\tilde{H}_n(S^1) = 0$ for $n \geq 2$)

Assume statement for $l < k$

Then $\tilde{H}_{n-1}(\partial D^k) = \tilde{H}_{n-1}(S^{k-1}) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k, 0 \end{cases}$

So $H_n(S^k) \cong H_n(D^k, \partial D^k) \cong H_{n-1}(\partial D^k)$

$$= \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k, 0 \end{cases}$$

$n = 0$ ✓ $\Rightarrow$ done.

Note: by induction, we can see that a generator of

$$\tilde{H}_n(S^n) \cong H_n(D^n, \partial D^n) \cong \mathbb{Z}$$

is represented by the relative n-cycle

$\Delta_1^n \xrightarrow{\simeq} D^n$

or by the n-cycle $\Delta_1^n - \Delta_2^n$ where

$$S^n = \frac{\Delta_1^n \sqcup \Delta_2^n}{\partial \Delta_1^n = \partial \Delta_2^n}.$$

for $H_n(\Delta^n, \partial \Delta^n) \cong H_n(S^n, \Delta_2^n) \cong H_n(S^n)$

by excision. ?? long exact sequences. pg 125 HATCHER

9 Mayer - Vietoris sequence

"Van - Kampen for H_*"

Let X be topological space and $A, B \subset X$ with $\mathring{A} \cup \mathring{B} = X$. There is a long exact sequence

$$H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B)$$

$$[\sigma] \mapsto ([\sigma], -[\sigma])$$

$$([\sigma], [\mu]) \to [\sigma + \mu]$$

$$[\underset{C_n(A)}{\sigma} + \underset{C_n(B)}{\mu}] \mapsto [\partial \sigma] = -[\partial \mu].$$

Proof: Apply locality theorem $\mathcal{U} = \{A, B\}$

$C^{\mathcal{U}}(X) \hookrightarrow C(X)$ which induces isomorphism

and consider short exact sequence of chain complexes

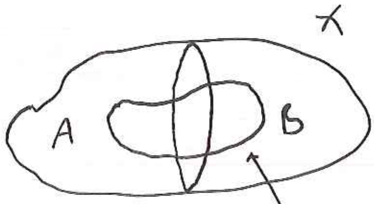
$$0 \to C_\cdot(A \cap B) \to C_\cdot(A) \oplus C_\cdot(B) \to C^{\mathcal{U}}(X) \to 0$$

$$\sigma \mapsto (\sigma, -\sigma)$$

$$(\sigma, \mu) \mapsto \sigma + \mu$$

Now apply snake lemma.

see problem sheet to be a bit more precise



cycle in X, which is sum of chains in A + chains in B. See boundaries are the same with opposite orientation.

Ex. The cone on X is the quotient space

$$CX = \frac{X \times [0,1]}{\sim} \qquad (x,s) \sim (y,t) \text{ iff } s = t = 1. \text{ or } (x,s) = (y,t)$$

The suspension on X is the quotient space

$$\Sigma X = \frac{X \times [0,1]}{\sim} \qquad (x,s) \sim (y,t) \text{ iff } s = t = 0 \text{ or } 1 \text{ or } (x,s) = (y,t)$$

e.g. $CS^{k-1} \cong D^k$

$\Sigma S^{k-1} \cong S^k$

Note $CX \simeq *$.

Apply M-V to $X = \Sigma X$, $A = \frac{X \times [0, \frac{2}{3}]}{\sim} \subset \Sigma X$

$B = \frac{X \times [\frac{1}{3}, 1]}{\sim} \subset \Sigma X$

To get $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$ for all $n \geq 1$

(N.B. $A \cap B = \frac{[\frac{1}{3}, \frac{2}{3}]}{\sim} \cong X$)

10 Degree of a map $f: S^n \to S^n$

Recall $\tilde{H}_n(S^n) = \mathbb{Z}$. Hence, f_* in dimension n is just multiplication by an integer, $\deg(f)$, the degree of f.

N.B. i) $\deg(\mathrm{id}_{S^n}) = 1$
ii) $\deg(f \circ g) = \deg f \deg g$
iii) $f \simeq g \Rightarrow \deg f = \deg g$

$\deg : C(S^n) \to \mathbb{Z}$ is a monoid homomorphism

Ex: i) f is the induced map by a reflection in $\mathbb{R}^n < \mathbb{R}^{n+1}$:

a generator for $H_n(S^n)$ is $\Delta_1^n - \Delta_2^n$ and

$f_\#(\Delta_1^n - \Delta_2^n) = (\Delta_2^n - \Delta_1^n)$. So $\deg f = -1$.

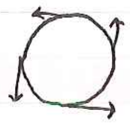
2) f is antipodal map $S^n \to S^n$
$x \mapsto -x$

f is induced by $-I : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$

$$-I = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & \ddots \; 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$$

is composition of $(n+1)$-reflections $\Rightarrow \deg f = (-1)^{n+1}$

Application: S^n has a continuous vector field of non-zero tangent vectors iff n is odd.

S^1:  S^2: ...

Hairy-Ball Theorem: you cannot comb a hairy ball.

Proof:
Suppose $v: S^n \to \mathbb{R}^{n+1}$ is s.t. $v(x) \perp x$ and $v(x) \neq 0$ for all $x \in S^n$.

Consider $F : S^n \times [0,1] \to S^n$ given by
$(x,t) \mapsto (\cos \pi t)x + (\sin \pi t)\dfrac{v(x)}{\|v(x)\|}$

N.B. $F(x,0) = x$ and $F(x,1) = -x$
so F is a homotopy from id to antipodal map.

So $1 = \deg(\mathrm{id}) = \deg(\text{antipodal}) = (-1)^{n+1}$

and so n is odd.

Conversely, if n is odd, $n+1 = 2k$ is even.

$$v : S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^{2k} \longrightarrow \mathbb{R}^{2k}$$

$$x = (x_1, \dots, x_{2k}) \longmapsto (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}) \neq 0$$

and $v(x) \perp x$. □

Let $f(x) = y$ and $U \ni x$, $V \ni y$ be open neighbourhoods of x and y s.t.

$$f : (U, U \setminus \{x\}) \longrightarrow (V, V \setminus \{y\})$$

ie the only inverse image of y in U is x.
From the relative long exact sequence and excision we have $(n > 0)$

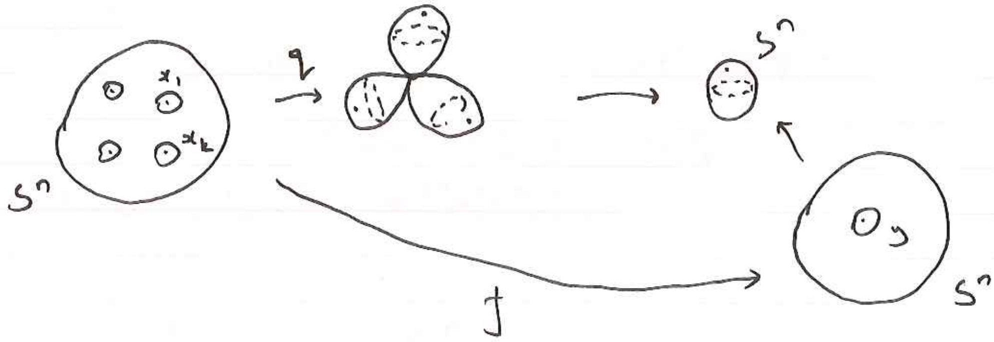
$$\begin{tikzcd} \mathbb{Z} = H_n S^n \cong H_n(U, U \setminus \{x\}) \arrow[r] \arrow[d, phantom, "\wr\|"] & H_n(V, V \setminus \{y\}) \cong H_n S^n = \mathbb{Z} \arrow[d, phantom, "\wr\|"] \\ H_n(D^n, D^n \setminus \{0\}) & H_n(D^n, D^n \setminus \{0\}) \end{tikzcd}$$

$(f|_x)_*$ is multiplication by an integer $\deg f_x$, the local degree of f at x.

Proposition: Assume $f^{-1}(y) = \{x_1, \dots, x_k\}$.

Then $\deg f = \sum_{i=1}^{k} \deg f|_{x_i}$

Proof:



Suppose $f: S^n \to S^n$ $(n>0)$ has the property that for

some point $y \in S^n$, the preimage $f^{-1}(y)$ consists of only finitely many points, say $x_1, \cdots, x_m$. Let $U_1, \cdots, U_m$ be disjoint neighborhoods of these points, mapped by f into a neighborhood V of y. Then $f(U_i - x_i) \subset V - y$ for each i, and we have a commutative diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\approx"'] \arrow[d, "k_i"] & H_n(V, V-y) \arrow[d, "\approx"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\approx"] \arrow[u, "j"] & H_n(S^n) \arrow[u, "\approx"']
\end{tikzcd}$$

where all the maps are the obvious ones, in particular k_i and p_i are induced by inclusions. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. Via these four isomorphisms, the top two groups in the diagram can be identified with $H_n(S^n) \approx \mathbb{Z}$, and the top homomorphism f_* becomes multiplication by an integer called the **local degree** of f at x_i, written $\deg f \,|\, x_i$.

For example, if f is a homeomorphism, then y can be any point and there is only one corresponding x_i, so all the maps in the diagram are isomorphisms and $\deg f \,|\, x_i = \deg f = \pm 1$. More generally, if f maps each U_i homeomorphically onto V, then $\deg f \,|\, x_i = \pm 1$ for each i. This situation occurs quite often in applications, and it is usually not hard to determine the correct signs.

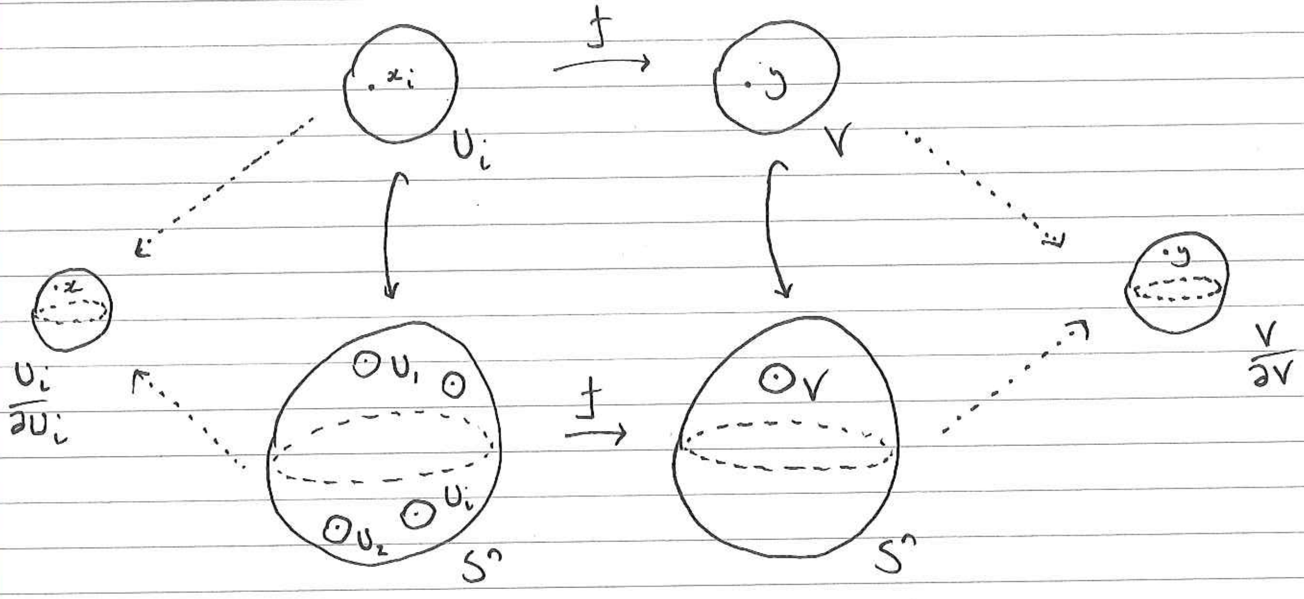
Here is the formula that reduces degree calculations to computing local degrees:

Proposition 2.30. $\deg f = \sum_i \deg f \,|\, x_i$.

Proof: By excision, the central term $H_n(S^n, S^n - f^{-1}(y))$ in the preceding diagram is the direct sum of the groups $H_n(U_i, U_i - x_i) \approx \mathbb{Z}$, with k_i the inclusion of the
* i^{th} summand. Since the upper triangle commutes, the projections of this direct sum onto its summands are given by the maps p_i. Identifying the outer groups in the diagram with $\mathbb{Z}$ as before, commutativity of the lower triangle says that $p_i j(1) = 1$, hence $j(1) = (1, \cdots, 1) = \sum_i k_i(1)$. Commutativity of the upper square says that the middle f_* takes $k_i(1)$ to $\deg f \,|\, x_i$, hence $\sum_i k_i(1) = j(1)$ is taken to $\sum_i \deg f \,|\, x_i$. Commutativity of the lower square then gives the formula $\deg f = \sum_i \deg f \,|\, x_i$. □

Example 2.31. We can use this result to construct a map $S^n \to S^n$ of any given degree, for each $n \geq 1$. Let $q: S^n \to \bigvee_k S^n$ be the quotient map obtained by collapsing the complement of k disjoint open balls B_i in S^n to a point, and let $p: \bigvee_k S^n \to S^n$ identify all the summands to a single sphere. Consider the composition $f = pq$. For almost all $y \in S^n$ we have $f^{-1}(y)$ consisting of one point x_i in each B_i. The local degree of f at x_i is ± 1 since f is a homeomorphism near x_i. By precomposing p with reflections of the summands of $\bigvee_k S^n$ if necessary, we can make each local degree either $+1$ or -1, whichever we wish. Thus we can produce a map $S^n \to S^n$ of degree $\pm k$.

* This shows $p_i k_i = id$. To see $p_j k_i = 0$ when $i \neq j$ note that the inclusion $(U_i, U_i - x_i) \to (S^n, S^n - x_j)$ factors through $(U_i, U_i - x_i) \to (U_i, U_i)$. Hence, $H_n(U_i, U_i - x_i) \to H_n(S^n, S^n - x_j)$ factors through $H_n(U_i, U_i - x_i) \to H_n(U_i, U_i) = 0$.



The maps to $\frac{U_i}{\partial U_i}$, $\frac{V}{\partial V}$ do not really exist.
They give intuition.

The proposition follows from commutativity of groups:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] \arrow[dd, bend right=60, "\text{diagonal map}"'] & H_n(S^n) \arrow[d, "\cong \text{ l.e.s.}"] \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & H_n(S^n, S^n \setminus \{y\}) \\
\bigoplus_{i=1}^{k} H_n(U_i, U_i \setminus \{x_i\}) \arrow[u, "\cong \text{ excision}"] \arrow[r, "\sum_i (f|_{x_i})_*"] & H_n(V, V \setminus \{y\}) \arrow[u, "\cong \text{ excision}"']
\end{tikzcd}$$

$$(S^n, S^n \setminus \{x_i\}) \longleftrightarrow (U_i, U_i \setminus \{x_i\})$$
$$\uparrow \simeq$$
$$(S^n, S^n \setminus U_i)$$

$$(U, U \setminus \{x\}) \longleftrightarrow (U, \partial U)$$

?? see print out...

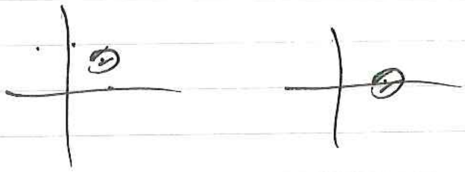
Ex: $f : S^2 = \mathbb{C} \cup \{\infty\} \longrightarrow S^2$ given by $f(z) = z^n$

Put $y = 1$

$$f^{-1}\{y\} = f^{-1}\{1\} = \left\{ e^{\frac{2\pi i}{n} k} : k = 1, \dots, n \right\}$$

$$\deg f\big|_{e^{\frac{2\pi i k}{n}}} = 1$$

$$\deg f = n$$



locally, orientation preserving homeomorphism

For an arbitrary polynomial $p(z)$ of degree n,

$$p(z) \sim z^n$$
$$\shortparallel$$
$$a_n z^n + \dots + a_0$$
$$\sim a_n z^n \sim z^n$$
$$\Rightarrow \deg p(z) = n$$

Also, we see if $f(z) = z^n$ $\deg f|_0 = \deg f = n$

Definition : CW - complex

Construct inductively

1) X^0 a discrete set

2) X^n is built from X^{n-1} by attaching n-cells :

Let $\{(D^n_\alpha, \phi_\alpha)\}$ be a collection of n-disks

together with attaching maps (cont.)

$$\phi_\alpha : \partial D^n_\alpha = S^{n-1} \longrightarrow X^{n-1}$$

$$X^n = X^{n-1} \amalg_\alpha D^n_\alpha \Big/ \sim \quad \text{where } x \sim \phi_\alpha(x) \text{ for } x \in \partial D^n_\alpha .$$

3) $X = \bigcup_{n \in \mathbb{N}} X^n$

is given the weak Topology

($A \subset X$ is open iff $A \cap X_n \subset X_n$ is open for all $n \in \mathbb{N}$)

interior is X^n

$\mathring{D}^n_\alpha = e^n_\alpha \subset X$ is called an n-cell

see HATCHER

$X^n \subset X$ is called the n-skeleton, $X^{-1} = \phi$.

Ex : any simplicial set

any Δ - complex

$X^0 = pt.$

$X^1 =$ 

$X^2 =$  $\amalg D^2$ $\quad \phi : \partial D^2 = S^1 \longrightarrow S^1$

$z \longmapsto z^3$

deg 3 map.

Each (X^m, X^{m-1}) is good and $\dfrac{X^m}{X^{m-1}} = \bigvee_\alpha S^m_\alpha$

e^m_α an n-cell.

Define for each m-cell e^m_α and $(m-1)$-cell e^{m-1}_β

$$d_{\alpha\beta} = \deg \left(\partial D^m_\alpha = S^{m-1} \xrightarrow[\phi_\alpha]{} X^{m-1} \xrightarrow[q]{} \frac{X^{m-1}}{X^{m-2}} = \bigvee_\alpha S^{m-1}_\alpha \rightarrow S^{m-1} = \frac{D^{m-1}_\beta}{\partial D^{m-1}_\beta} \right)$$

Define $C_m^{CW}(X) := H_m(X^m, X^{m-1})$ = free abelian group on the n-cells

and $d : C_m^{CW}(X) \longrightarrow C_{m-1}^{CW}(X)$

$$e_\alpha^m \longmapsto \sum_\beta d_{\alpha\beta} e_\beta^{m-1}$$

(obvious thing when $n=1$, as this formula does not hold... $d_{\alpha\beta}$?)

Lemma: $d^2 = 0$

Definition: Cellular Homology $H_n^{CW}(X) := H_n(C^{CW}(X), d)$

Ex:

$$\begin{array}{ccccc} C_2^{CW}(X) & \xrightarrow{d} & C_1^{CW}(X) & \xrightarrow{d} & C_0^{CW}(X) \\ \| & & \| & & \| \\ H_2(X^2, X^1) & & H_1(X^1, X^0) & & H_0(X^0) \\ \| & & \| & & \| \\ H_2(S^2) = \mathbb{Z} & & H_1(S^1) = \mathbb{Z} & & H_0(pt) = \mathbb{Z} \\ = \langle e_\alpha^2 \rangle & & = \langle e_\beta^1 \rangle & & = \langle e_\gamma^0 \rangle \end{array}$$

$d : C_2^{CW}(X) \to C_1^{CW}(X) \quad = \quad d_{\alpha\beta} = 3$

$d : C_1^{CW}(X) \longrightarrow C_0^{CW}(X) \quad = \quad d_{\beta\gamma} = 0$

$$H_n^{CW}(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \dfrac{\mathbb{Z}}{3\mathbb{Z}} & n = 1 \\ 0 & n \geq 2. \end{cases}$$

Note: For simplicial complexes, the attaching maps are of degree 1. So then a Δ-complex is a simplicial complex

$$H_m^{CW}(X) = H_m^\Delta(X) \quad \text{by definition.}$$

Proof of lemma: Consider the l.e.s. in relative homology

P.T.O.

CELLULAR HOMOLOGY.

Lemma: If X is a CW complex, then:

a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.

b) $H_k(X^n) = 0$ for $k > n$.

c) $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Proof: a) (X^n, X^{n-1}) is a good pair and $\frac{X^n}{X^{n-1}}$ is a wedge sum of n-spheres, one for each n-cell of X.

b) The l.e.s. of the pair (X^n, X^{n-1}) contains the segments

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}),$$

so by a), $k \neq n, n-1 \Rightarrow H_k(X^{n-1}) \cong H_k(X^n)$. Hence $k > n \Rightarrow$
$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \dots \cong H_k(X^0) = 0$.

c) If $k < n$, $H_k(X^n) \cong H_k(X^{n+1}) \cong H_k(X^{n+2}) \cong \dots \cong H_k(X^{n+m})$ for all $m \geq 0$, which suffices when X is finite-dimensional, noting the isomorphisms are induced by inclusion.

When X is infinite-dimensional, we use that a singular chain in X has compact image and so meets only finitely many cells of X.

Hence: A k-cycle in X is a cycle in some X^{n+m} $(m \geq 0)$ and by the finite dimensional case the cycle is homologous to a cycle in X^n, showing surjectivity of $i_* : H_k(X^n) \to H_k(X)$.

Also: If a k-cycle in X^n bounds a chain in X, this chain lies in some X^{n+m} $(m \geq 0)$ and by the finite-dimensional case, the cycle bounds a chain in X^n, showing injectivity of i_*.

So, the l.e.s. for the pair (X^n, X^{n-1}) gives an exact sequence:

$$0 \to H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\delta_n} H_{n-1}(X^{n-1}) \to H_{n-1}(X^n) \cong H_{n-1}(X) \to 0$$

and portions of the l.e.s. for the pairs (X^{n+1}, X^n), (X^n, X^{n-1}), (X^{n-1}, X^{n-2}) fit into a diagram:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[r] & 0 & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\dots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\delta_{n+1}"] \arrow[rr, "d_{n+1}"'] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\delta_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \dots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

where $d_n = j_{n-1} \delta_n$.

Define $\quad C_n^{CW}(X) := H_n(X^n, X^{n-1})$ = free abelian group on the n-cells, and let $d_n = j_{n-1}\delta_n : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ be as above.

Then $\quad d_{n-1}d_n = j_{n-2}\delta_{n-1}j_{n-1}\delta_n = j_{n-2} \circ \delta_n = 0$ and the cellular homology of X is $H_n^{CW}(X) := H_n(C_\cdot^{CW}(X))$.

Theorem : $\quad H_n^{CW}(X) \cong H_n(X)$.

Proof : $\quad H_n(X)$ can be identified with $H_n(X^n)/\operatorname{im}\delta_{n+1}$. Since j_n is injective, it maps $\operatorname{im}\delta_{n+1}$ isomorphically onto $\operatorname{im}(j_n\delta_{n+1}) = \operatorname{im} d_{n+1}$, and $H_n(X^n)$ isomorphically onto $\operatorname{im} j_n = \ker\delta_n$. Since j_{n-1} is injective $\ker\delta_n = \ker d_n$. Thus j_n induces an isomorphism $H_n(X^n)/\operatorname{im}\delta_{n+1} \xrightarrow{\cong} \ker d_n/\operatorname{im} d_{n+1}$.

Theorem : $\quad d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1} \quad (n>1)$ where $d_{\alpha\beta} = \deg\left(S_\alpha^{n-1} \xrightarrow{\phi_\alpha} X^{n-1} \to X^{n-1}/X^{n-1}\setminus\{e_\beta^{n-1}\}\right)$ and ϕ_α is the characteristic map of the n-cell e_α^n.

Proof :

$$\begin{tikzcd}
H_n(D_\alpha^n, \partial D_\alpha^n) \arrow[r, "\delta", "\cong"'] \arrow[d, "\Phi_{\alpha *}"'] & \tilde{H}_{n-1}(\partial D_\alpha^n) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\phi_{\alpha *}"] & \tilde{H}_{n-1}(S_\beta^{n-1}) \cong H_{n-1}(D_\beta^{n-1}, \partial D_\beta^{n-1}) \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta_n"] \arrow[dr, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \tilde{H}_{n-1}\left(\frac{X^{n-1}}{X^{n-2}}\right) \arrow[u, "q_{\beta *}"'] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur, leftrightarrow, "\cong"'] &
\end{tikzcd}$$

Hatcher.

$$H_m\left(\frac{X^{m+1}}{X^m}\right) = 0$$

$$H_{m+1}(X^{m+1}, X^m) \xrightarrow{\delta} H_m(X^m) \to H_m(X^{m+1}) \to \underbrace{H_m(X^{m+1}, X^m)}_{\cong H_m\left(\frac{X^{m+1}}{X^m}\right) = 0}$$

$$\underbrace{0 = H_m(X^{m-1})}\to H_m(X^m) \xrightarrow{q} H_m(X^m, X^{m-1}) \xrightarrow{\delta} H_{m-1}(X^{m-1})$$

$$\underbrace{0 = H_{m-1}(X^{m-2})}_{\text{by later}} \to H_{m-1}(X^{m-1}) \xrightarrow{q} H_{m-1}(X^{m-1}, X^{m-2}) \xrightarrow{\delta} H_{m-2}(X^{m-2})$$

$$H_{m-1}(X^{m-1}, X^{m-2}) \cong \tilde{H}_{m-1}\left(\frac{X^{m-1}}{X^{m-2}}\right)$$

By Q3 on sheet 3 (I think)

$$\delta[e^n_\alpha] = [\phi_\alpha(\partial D^n_\alpha)]$$

So $d = q \circ \delta$

Hence $d^2 = (q \circ \delta) \circ (q \circ \delta) = q \circ (\delta \circ q) \circ \delta = 0$
as $\delta \circ q$ is zero in middle l.e.s.

Ex: S^m $\quad X^0 = pt \quad (= X^1 = X^2 = \dots = X^{m-1})$

$$S^m = X^m = pt \cup_\alpha D^m \qquad \alpha : \partial D^m = S^{m-1} \to pt.$$

$$0 \to C^{CW}_m(X) = \mathbb{Z} \to C^{CW}_{m-1}(X) = 0 \to 0 \to 0$$
$$\to \dots \to C^{CW}_1(X) = 0 \to C^{CW}_0(X) = \mathbb{Z} \to 0$$

$$m \geq 2 \Rightarrow H^{CW}_n(S^m) = \begin{cases} \mathbb{Z} & n = 0, m \\ 0 & n \neq 0, m \end{cases}$$

$$m = 1 \quad H^{CW}_n(S^1) = H_n(\dots \to 0 \to \mathbb{Z} \xrightarrow{d=0} \mathbb{Z} \to 0)$$
$$= \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n \neq 0, 1. \end{cases}$$

Theorem: For any CW-complex $H^{CW}_m(X) \cong H_m(X)$

(in particular $H^\Delta_m(X) \cong H_m(X)$
for any simplicial complex, and Δ-complex)

Proof: $H_m(X) \cong H_m(X^{m+1})$ (from the relative l.e.s.)

$$\left[H_{m+1}(X) \to H_m(X^{m+1}) \to H_m(X) \to H_{m-1}(X^{m+1}) \right]$$

$$\left[H_{m+1}(X, X^{m+1}) \to H_m(X^{m+1}) \to H_m(X) \to H_m(X, X^{m+1}) \right]$$

Lemma: $H_m(X) = \lim_{n\to\infty} H_m(X^n)$

Proof: a compact set in X is contained in X^n for some n; as images of compact sets are compact

$$C.(X) = \lim_{n\to\infty} C.(X^n)$$

If c is a cycle in $C.(X^n)$ it will also be in $C.(X)$.

If there exists $b \in C.(X)$ s.t. $\partial b = c$ then again by compactness

$$b \in C.(X^{\tilde{n}}) \text{ for some } \tilde{n}$$

and hence $[c] = 0$ in $H_*(\tilde{X}^n)$

So $\lim_{k\to\infty} H_m(X^k) = H_m(X)$

Lemma: 1) $H_m(X^k) = 0$ for $m > k$
2) $H_m(X^k) = H_m(X)$ for $m < k$

Proof: Consider the l.e.s. in rel H_*.

$$H_{m+1}(X^k, X^{k-1}) \xrightarrow{\delta} H_m(X^{k-1}) \to H_m(X^k) \to H_m(X^k, X^{k-1})$$

As $\frac{X^k}{X^{k-1}} \simeq \vee S^k$, if $m \neq k, k-1$ $H_m(X^{k-1}) \cong H_m(X^k)$

for 1), $m > k$, $H_m(X^k) \cong H_m(X^{k-1}) \cong \dots \cong H_m(X^0) = 0$.

for 2), $m < k$, $H_m(X^k) \cong H_m(X^{k+1}) \cong \dots \cong H_m(X)$ by Lemma.

Theorem: For any CW complex X, $H_m^{CW}(X) \cong H_m(X)$.

Proof: refer to last major diagram.

$$\ker d_m = \ker \delta_m = \operatorname{im} q_* \cong H_m(X^m)$$
$$\operatorname{im} d_{m+1} \cong \operatorname{im} \delta_{m+1}$$

$$H_m^{CW}(X) = \frac{\ker d_m}{\operatorname{im} d_{m+1}} \cong \frac{H_m(X^m)}{\operatorname{im} \delta_{m+1}} \cong H_m(X^{m+1}) \cong H_m(X)$$

Ex: $\mathbb{RP}^n$ n^{th} projective space

$(= \text{lines in } \mathbb{R}^{n+1})$

$= \dfrac{S^n}{x \sim -x}$.

$\mathbb{RP}^0 = * = \dfrac{S^0}{1 \sim -1}$

$\mathbb{RP}^1 =$  $= \mathbb{RP}^0 \cup_\alpha D^1 \qquad \alpha : \partial D^1 = S^0 \to \mathbb{RP}^0 = \dfrac{S^0}{\sim}$

identify

$\mathbb{RP}^2 =$  $= \mathbb{RP}^1 \cup_\alpha D^2 \qquad \alpha : \partial D^2 = S^1 \to \mathbb{RP}^1 = \dfrac{S^1}{\sim}$

$\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_\alpha D^n \qquad \alpha : \partial D^n = S^{n-1} \to \mathbb{RP}^{n-1} = \dfrac{S^{n-1}}{\sim}$

$\mathbb{RP}^n$ has a cell decomposition in each dimension

$$d = d_m = \deg\left(S^{m-1} \xrightarrow[\alpha]{} \mathbb{RP}^{m-1} \xrightarrow[q]{} \frac{\mathbb{RP}^{m-1}}{\mathbb{RP}^{m-2}} \cong S^{m-1}\right)$$

$$q \circ \alpha \big|_{\mathring{\Delta}_+}, \quad q \circ \alpha \big|_{\mathring{\Delta}_-}$$

are homeomorphisms onto their image which are related to each other by the antipodal map

$$d = \deg(q \circ \alpha) = \deg(\mathrm{id}) + \deg(\text{antipodal map})$$
$$= 1 + (-1)^{(m-1)+1} = 1 + (-1)^m$$

$$\Rightarrow C^W(\mathbb{RP}^n) = \begin{cases} 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \dots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 & \text{odd} \\ 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \dots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 & \text{even} \end{cases}$$

n

$\Rightarrow \quad C^W(\mathbb{R}P^n) = \begin{cases} 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \dots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 & \text{odd} \\ 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \dots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 & \text{even} \end{cases}$

$\uparrow$
n

$$\Rightarrow \quad H_m(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & m = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}} & m = \text{odd and } 1 \leq m < n \\ \mathbb{Z} & m = n \text{ odd} \\ 0 & \text{o/w} \end{cases}$$

Note : $H_n(\mathbb{R}P^n) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}$

With coefficients in $\mathbb{Z}_2$ we don't see this.

12 COHOMOLOGY

Let $(C_\cdot, \partial_\cdot)$ be a chain complex of abelian groups (or R-modules). Define

n-cochains $\quad C^n := \mathrm{Hom}(C_n, \mathbb{Z}) \quad$ (or $\mathrm{Hom}_R(C_n, R)$)

co-boundary $\quad \partial^*_n : C^{n-1} \longrightarrow C^n$
$\qquad\qquad \varphi \longmapsto \varphi \circ \partial_n$

n-cocycles $\quad Z^n := \ker(\partial^*_{n+1}) \subset C^n$
n-coboundaries $\quad B^n := \mathrm{im}(\partial^*_n) \subset C^n$

n^{th} cohomology $\quad H^n(C_\cdot, \partial_\cdot) := \frac{Z^n}{B^n}$

When $C_\cdot$ is $\Delta_\cdot(X)$, $C_\cdot(X)$, $C_\cdot^{CW}(X)$ we speak of simplicial, singular, or cellular cohomology of X. $\cong H^n(X)$.

Contravariant Functoriality : $f : X \to Y$
$\Rightarrow f_\# : C_\cdot(X) \to C_\cdot(Y)$

$\Rightarrow \quad f^{\#} = f_{\#}^{\ *} : C^*(Y) \to C^*(X) \qquad \varphi \mapsto \varphi \circ f_{\#}$

$\Rightarrow \quad f^* : H^n(Y) \to H^n(X)$

(check is a chain map, easy...)

Homotopy invariance: $f \simeq g : X \to Y$ homotopic

$\Rightarrow \quad f_{\#} - g_{\#} = \partial P - P\partial$

$\Rightarrow \quad f^{\#} - g^{\#} = P^*\partial^* - \partial^* P$

$\Rightarrow \quad f^* = g^* : H^n(Y) \to H^n(X)$

Lemma: If $0 \to A_\cdot \to B_\cdot \to C_\cdot \to 0$ is a short exact sequence of chain complexes, then we get a short exact sequence of cochain complexes

$$0 \leftarrow A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$$

provided the cochain groups are free.

Proof: Note that this is a sequence of cochain complexes.

As C_n is free, there exists a splitting

$$0 \to A_n \xrightarrow{i} B_n \underset{s}{\rightleftarrows} C_n \to 0$$

see HATCHER UCT proof

So $\quad A_n \oplus C_n \xrightarrow[i+s]{\cong} B_n$

So $\quad \underbrace{(A_n \oplus C_n)^* \cong}_{\text{naturally}} A_n^* \oplus C_n^* \xleftarrow[i^*+s^*]{\cong} B_n^*$

Hence,

$$0 \leftarrow A_n^* \xleftarrow{i^*} B_n^* \underset{s^*}{\leftrightarrows} C_n^* \leftarrow 0 \quad \text{is short exact.}$$

Corollary: For any pair of spaces (X,A) there is a l.e.s.

$$\cdots \leftarrow H^1(X,A) \leftarrow H^0(A) \leftarrow H^0(X) \leftarrow H^0(X,A) \leftarrow 0$$

Corollary: For $X = \mathring{A} \cup \mathring{B}$ there is a long exact sequence

$$\cdots \leftarrow H^1(X) \leftarrow H^0(A \cap B) \leftarrow H^0(A) \oplus H^0(B) \leftarrow H^0(X) \leftarrow 0.$$

Excision $\quad Z \subset \mathring{A} \subset X, \quad H^n(X,A) \xrightarrow{\cong} H^n(X \setminus Z, A \setminus Z)$ induced by inclusion

Ex: The cellular chain complex of $\mathbb{RP}^3$ is

$$0 \to \underset{3}{\mathbb{Z}} \xrightarrow{0} \underset{2}{\mathbb{Z}} \xrightarrow{2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to 0$$

The dual of this is $\left(\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & \cong & \mathbb{Z} \\ \varphi & \mapsto & \varphi(1) \end{array}\right)$

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

Dual of 2 is
$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & \leftarrow & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \\ \varphi \circ 2 & \leftarrow\!\shortmid & \varphi \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ 2\varphi(1) = (\varphi\circ 2)(1) & \leftarrow\!\shortmid & \varphi(1) \end{array}$$

Hence,
$$H^n(\mathbb{RP}^3) \cong \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1 \\ \mathbb{Z}_2 & n=2 \\ \mathbb{Z} & n=3 \\ 0 & n\geq 4 \end{cases}$$

Constrast,
$$H_n(\mathbb{RP}^3) \cong \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}_2 & n=1 \\ 0 & n=2 \\ \mathbb{Z} & n=3 \\ 0 & n\geq 4 \end{cases}.$$

Universal Coefficient Theorem

1) Let X be a CW-complex with finitely many cells in every dimension (X is s.t.b. of finite type). Then
$$H^n(X;\mathbb{Z}) = H^n(X) \cong H_n(X)/\mathrm{Tor}(H_n(X)) \oplus \mathrm{Tor}(H_{n-1}(X))$$

(Here $\mathrm{Tor}(A)$ denotes the (maximal) torsion subgroup of the finitely generated abelian group A: $A \cong \mathbb{Z}^k \oplus \mathrm{Tor}(A)$)

2) Let X be any space. Then
$$H^n(X,\mathbb{F}) \cong H_n(X,\mathbb{F})^* = \mathrm{Hom}_{\mathbb{F}}(H_n(X,\mathbb{F}),\mathbb{F}).$$

Proof: i) See problem sheet 6.1

2) Let $C.$ be a chain complex

Then

$$\begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & Z_n & \hookrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \to & 0 \\ & & \uparrow 0 & & \uparrow \partial_{n+1} & & \uparrow 0 & & \\ 0 & \to & Z_{n+1} & \hookrightarrow & C_{n+1} & \to & B_n & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \end{array}$$

is a short exact sequence of chain complexes. By the lemma, this gives rise to a short exact sequence of cochain complexes.

$$0 \leftarrow Z^* \leftarrow C^* \leftarrow B^{*-1} \leftarrow 0.$$

BAD NOTATION: Here Z^n means Z_n^*, not what it was previously defined as

By the Snake Lemma, we get a l.e.s.

$$\leftarrow B^n \xleftarrow{\delta} Z^n \leftarrow H^n(C.) \leftarrow B^{n-1} \xleftarrow{\delta} Z^{n-1}$$

Under these identifications $\delta = i^*$ induced by $B_{n-1} \hookrightarrow Z_{n-1}$.

So there are s.e.s.

$$0 \leftarrow \ker i_* \leftarrow H^n(C.) \leftarrow \operatorname{coker} i_* \leftarrow 0$$

$$\ker i_n^* = \{ \varphi \in Z_n : \varphi|_{B_n} = 0 \}$$
$$= \left\{ \varphi \in \left(\frac{Z_n}{B_n}\right)^* \right\} = \left(\frac{Z_n}{B_n}\right)^* = (H_n(C.))^*$$

See HATCHER UCT proof.

As we are working over a field, $\operatorname{coker} i^* = 0$ as any $\varphi \in B^{n-1} = B_{n-1}^*$ can be extended to a map

$$Z_{n-1} \to \mathbb{F}.$$

(So its in the image of the restriction map i_*).

$\left(\phi : A \to B \quad \operatorname{coker} \phi = \frac{B}{\operatorname{im} \phi} \right)$

12 Cup Products

Let X be a space, and $(C^\cdot(X), \partial^*)$ be its singular cochain complex. (similarly simplicial if X is a Δ-complex)

For $\varphi \in C^k(X)$ and $\psi \in C^l(X)$...
The cup product $\varphi \cup \psi \in C^{k+l}(X)$ is the cochain that maps $\sigma : \Delta^{k+l} \to X$ to

$$(\varphi \cup \psi)(\sigma) = \varphi\left(\sigma|_{[v_0, \dots, v_k]}\right) \psi\left(\sigma|_{[v_k, \dots, v_{k+l}]}\right)$$

Lemma: $\partial^*(\varphi \cup \psi) = \partial^* \varphi \cup \psi + (-1)^k \varphi \cup \partial^* \psi$

Proof: Let $\sigma : \Delta^{k+l+1} \to X$ be a singular simplex.

Then
$$\partial^*(\varphi \cup \psi)(\sigma) = (\varphi \cup \psi)(\partial \sigma)$$
$$= (\varphi \cup \psi)\left(\sum_{i=0}^{k+l+1} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{k+l+1}]}\right)$$
$$= \sum_{i=0}^{k} (-1)^i \varphi\left(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{k+1}]}\right) \psi\left(\sigma|_{[v_{k+1}, \dots, v_{k+l+1}]}\right)$$
$$+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi\left(\sigma|_{[v_0, \dots, v_k]}\right) \psi\left(\sigma|_{[v_k, \dots, \hat{v}_i, \dots, v_{k+l+1}]}\right)$$

$$= \sum_{i=0}^{k+1} \dots + \sum_{i=k}^{k+l+1} \qquad \text{(cancellation)}$$

$$= (\partial^* \varphi \cup \psi)(\sigma) + (-1)^k (\varphi \cup \partial^* \psi)(\sigma)$$

Corollary: The cup product induces an associative multiplication in cohomology

$$H^k(X) \times H^l(X) \xrightarrow{\cup} H^{k+l}(X)$$

ie $H^*(X) = \bigoplus_{k=0}^{\infty} H^k(X)$ is a graded ring.

Furthermore, the multiplication is graded commutative ie $[\varphi] \cup [\psi] = (-1)^{kl} [\psi] \cup [\varphi]$ (Proof of this part delayed)

Proof: Well-defined $\varphi \in Z^k$, $\psi \in Z^l$

$\Rightarrow$ $\partial^*(\varphi \cup \psi) = \partial^* \varphi \cup \psi + (-1)^k \varphi \cup \partial^* \psi = 0$

$\Rightarrow$ $\varphi \cup \psi \in Z^{k+l}$.

$\tilde{\varphi} = \varphi + \beta$, $\beta \in B^k$

$\Rightarrow$ $\tilde{\varphi} \cup \psi = \varphi \cup \psi + \beta \cup \psi$.

But $\beta = \partial^* \gamma$.

So $\tilde{\varphi} \cup \psi = \varphi \cup \psi + \partial^* \gamma \cup \psi + (-1)^k \gamma \cup \overset{0}{\overset{\shortparallel}{\partial^* \psi}}$

$= \varphi \cup \psi + \partial^*(\gamma \cup \psi)$

Hence $[\tilde{\varphi} \cup \psi] = [\varphi \cup \psi]$.

Unit: Let $\varepsilon \equiv 1 \in C^0(X)$, the map assigning 1 to any $\sigma : \Delta^0 \to X$.

For a 1-simplex σ:

$(\partial^* \varepsilon)(\sigma) = \varepsilon(\partial \sigma) = \varepsilon(\sigma|_{[v_1]}) - \varepsilon(\sigma|_{[v_0]})$

$= 1 - 1 = 0$

Hence $[\varepsilon] \in H^0(X)$

As $(\varphi \cup \varepsilon)(\sigma) = \varphi(\sigma|_{[v_0, \dots, v_k]}) \, \varepsilon(\sigma|_{[v_k]})$

we see that $[\varepsilon]$ is the unit in $H^*(X)$.

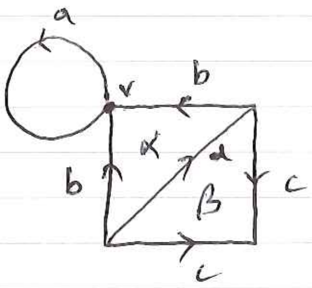
Proposition: Given $f : X \to Y$ then $f^\#(\varphi \cup \psi) = f^\#(\varphi) \cup f^\#(\psi)$

Hence $f^* : H^*(Y) \to H^*(X)$ is a map of rings.

Ex: $X = S^1 \vee S^2$

$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0, 1, 2 \\ 0 & n > 2 \end{cases}$$

$$\cong H^n(X) \cong \mathrm{Hom}(H_n(X), \mathbb{Z})$$



$H_0(X) \cong \langle [v] \rangle$

$H_1(X) \cong \langle [a] \rangle$

$H_2(X) \cong \langle [\alpha - \beta] \rangle$

Cochain representation for $[a]^*$

$$\begin{aligned} a &\mapsto 1 \\ b &\mapsto 0 \\ c &\mapsto 0 \\ d &\mapsto 0 \end{aligned}$$

Cochain representation for $[\alpha - \beta]^*$

$$\begin{aligned} \alpha &\mapsto 1 \\ \beta &\mapsto 0 \end{aligned}$$

$$H^0 \times H^k \xrightarrow{\cup} H^k$$
$$(n, x) \longmapsto nx$$

$H^l \times H^k \underset{0}{\xrightarrow{\cup}} H^{l+k}$ when $l + k > 2$.

Only interesting cup product

$$H^1 \times H^1 \longrightarrow H^2$$

$$([a]^*, [a]^*) \longrightarrow [a]^* \cup [a]^*$$

$$([a]^* \cup [a]^*)(\alpha) = [a]^*(d)\,[a]^*(b) = 0$$

$([a]^* \cup [a]^*)(\beta) = 0$ too!! $\Rightarrow [a]^* \cup [a]^* = 0$.

More generally $\tilde{H}^*(X \vee Y) \cong \tilde{H}^*(X) \oplus \tilde{H}^*(Y)$ as rings.

take cohomology
of extended chain complex
$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

13 Künneth Formula and Products

13a Algebraic Künneth Formula

If A and B are abelian groups (or R-modules) we define the tensor product $A \otimes B$ to be the abelian group generated by $\{a \otimes b\}_{a \in A, b \in B}$ subject to the relations

1) $(a + a') \otimes b = a \otimes b + a' \otimes b$
2) $a \otimes (b + b') = a \otimes b + a \otimes b'$

($\Rightarrow$ $na \otimes b = a \otimes nb = n(a \otimes b)$ for $n \in \mathbb{Z}$)

Note: Bilinear maps from $A \times B$ to C are the same as homomorphisms $A \otimes B \to C$

Ex: $\frac{\mathbb{Z}}{n\mathbb{Z}} \otimes \mathbb{Z} \cong \frac{\mathbb{Z}}{n\mathbb{Z}}$ $\qquad$ $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{nm}$

$\frac{\mathbb{Z}}{2\mathbb{Z}} \otimes \frac{\mathbb{Z}}{4\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$

If C and C' are two chain complexes of abelian groups then the tensor product is defined by

$$(C \otimes C')_n = \bigoplus_{k=0}^{n} C_k \otimes C'_{n-k}$$

and $\partial(c \otimes c') = \partial c \otimes c' \oplus (-1)^{\deg c} c \otimes \partial c'$

($\deg c = k$ means $c \in C_k$)

Check: $\partial^2(c \otimes c') = (-1)^{\deg(\partial c)} \partial c \otimes \partial c' + (-1)^{\deg c} \partial c \otimes \partial c' = 0$

- $Z_k(C) \otimes Z_{n-k}(C') \subset Z_n(C \otimes C')$
- $Z_k(C) \otimes B_{n-k}(C'),\ B_k(C) \otimes Z_{n-k}(C') \subset B_n(C \otimes C')$

Proposition: $C_\cdot \otimes C'_\cdot$ is a chain complex and there are natural maps

$$H_k(C) \otimes H_{n-k}(C') \longrightarrow H_n(C \otimes C')$$

P.T.O.

Künneth Theorem:
Assume C_k and $H_k(C)$ are free, for all k

Then $$\bigoplus_{k=0}^{n} H_k(C) \otimes H_{n-k}(C') \xrightarrow{\cong} H_n(C \otimes C')$$

13.6 Product Spaces

Let X and Y be cell complexes with cells $\{e_\alpha\}_{\alpha \in A}$ and $\{e_\beta\}_{\beta \in B}$ respectively. Then $X \times Y$ is a cell complex with cells $\{e_\alpha \times e_\beta\}_{\alpha \in A, \beta \in B}$. Then attaching map for each $e^k_\alpha \times e^{n-k}_\beta$ is given by

$$\varphi_{\alpha\beta} = \varphi_\alpha \times \Psi_\beta \cup \Phi_\alpha \times \psi_\beta$$

HATCHER NOTATION

$$: \partial(D^k \times D^{n-k}) = (\partial D^k \times D^{n-k}) \cup (D^k \times \partial D^{n-k}) \longrightarrow (X \times Y)^{n-1}$$

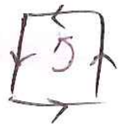
Point set topology problems ... HATCHER. or assume finite CW complexes

Recall $\chi(X \times Y) = \chi(X)\chi(Y)$

Proposition: $$C_\cdot^{CW}(X \times Y) \cong C_\cdot^{CW}(X) \otimes C_\cdot^{CW}(Y)$$

Proof: We only need to check that the sign in the boundary map is as desired!

HONEST!

Check it works for a square 

□

Künneth Theorem: Homology: If $H_k(X)$ is free for all $k \geq 0$

then $$\bigoplus_{k=0}^{n} H_k(X) \otimes H_{n-k}(Y) \xrightarrow{\cong} H_n(X \times Y).$$

X,Y CW complex

Cohomology: If $H^k(X)$ is free and finitely generated for all $k \geq 0$ then

$$\bigoplus_{k=0}^{n} H^k(X) \otimes H^{n-k}(Y) \xrightarrow{\cong} H^n(X \times Y)$$

$$a \otimes b \longmapsto p_X^*(a) \smile p_Y^*(b)$$

$$(p_X : X \times Y \to X, \ p_Y : X \times Y \to Y \text{ projections})$$

when X, Y are CW complexes we can define using cellular cohomology $\varphi \times \psi(e_\alpha \times e_\beta) = \varphi(e_\alpha) \times \psi(e_\beta) \ldots$

Ex: $X = S^i$, $Y = S^j$

Then $H_n(S^i \times S^j) \cong$ case $i \neq j$ $\begin{cases} \mathbb{Z} & n = 0, i, j, i+j \\ 0 & \text{if not} \end{cases}$

case $i = j$ $\begin{cases} \mathbb{Z} & n = 0, i+j \\ \mathbb{Z} \oplus \mathbb{Z} & n = i = j \\ 0 & \text{if not} \end{cases}$

Do universal theorem, followed by Kunneth formula to see $H^n(S^i \times S^j) \cong H_n(S^i \times S^j)$ and generators are

$$p_X^*(a), \ p_Y^*(b), \ p_X^*(a) \smile p_Y^*(b)$$

where a, b are generators for $H^i(S^i)$, $H^j(S^j)$.

Interpretation of the cup product:

$$H^k(X) \otimes H^{n-k}(X) \to H^n(X \times X) \xrightarrow{\Delta^*} H^n(X)$$

$$a \otimes b \longrightarrow a \smile b$$

where $\Delta : X \to X \times X$ is the diagonal map $x \mapsto (x, x)$.
From this interpretation it follows that

$$a \smile b = (-1)^{\deg a + \deg b}\, b \smile a.$$

14 Manifolds and Fundamental Class

A **manifold** of **dimension n** is a **Hausdorff** space M s.t. every point **$x \in M$** has an **open neighbourhood V_x** **homeomorphic to $\mathbb{R}^n$**. If **M** is **compact**, it is called **closed**.

Ex: S^n, $\mathbb{RP}^n = \dfrac{S^n}{\{\pm 1\}}$, $T^n = S^1 \times \dots \times S^1$ closed

$\mathbb{R}^n$, $GL_k\mathbb{R} \subset M_k\mathbb{R}$ not closed.

A **local orientation** of **M** is a **choice of generator**

$$\mu_x \in H_n(M, M - \{x\}) \cong H_n(V_x, V_x - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$$
$$\cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

(or $\cong H_n(M, M - V_x) \cong \tilde{H}_n(M/(M-V_x))$)

or $\cong H_n(M, M-B) \cong \tilde{H}_n(M/(M-B)) \cong \tilde{H}_n(S^n)$

where $B \subset \mathbb{R}^n$ ball finite avoiding problems with closure

An **orientation** of M is a **globally consistent** choice **$x \mapsto \mu_x$** so that

$$\begin{tikzcd} & H_n(M, M \setminus U) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_n(M, M - \{x\}) & & H_n(M, M - \{y\}) \\ \mu_x & & \mu_y \end{tikzcd}$$

(excision, 5 Lemma)

where $U \ni x, y$
$U \cong B \subset \mathbb{R}^n$

B: finite radius

N.B. This coincides with our notion of choosing an orientation: for each $x \in M$ we can make a consistent choice of up or down.

Ex: S^n is orientable
$\mathbb{RP}^2 = $ Möbius band $\cup\, D^2$ non-orientable

Theorem: If M is **orientable** of **dimension n**, then $H_n(M) \cong \mathbb{Z}$
If M is **not orientable** of dimension n, then $H_n(M) = 0$
and

$$H_n(M; \mathbb{Z}_2) \cong \mathbb{Z}_2. \quad \text{(both cases)}$$

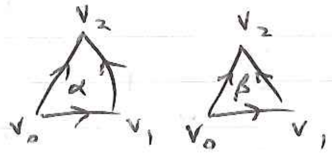
and closed, connected

Sketch: **Assume M is a Δ-set.**

A generator for $H_n(M)$ is "given by the manifold itself":

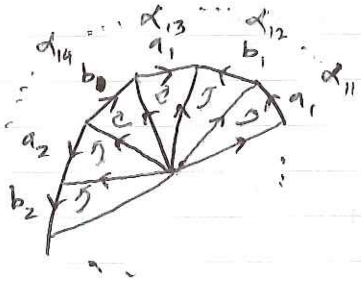
$\pm [M]$ is a sum of $(\pm)$ the n-simplices.

Ex: S^2:



$[S^2] = \alpha - \beta$
(or $\beta - \alpha$)

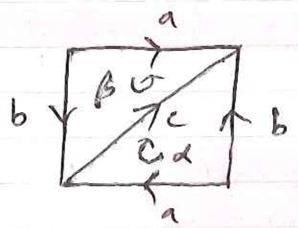
F_g:



$4g$-gon

$$[F_g] = \sigma_{11} + \sigma_{12} - \sigma_{13} - \sigma_{14} + \dots + \sigma_{g1} + \sigma_{g2} - \sigma_{g3} - \sigma_{g4}$$

Ex: $\mathbb{RP}^2$



$[\mathbb{RP}^2] = \alpha + \beta \pmod 2$

cycle in $C_2(\mathbb{RP}^2; \mathbb{Z}_2)$

Neither $\alpha + \beta$ or $\alpha - \beta$ is a cycle in $C_2(\mathbb{RP}^2; \mathbb{Z})$

$[M]$ is called the fundamental class of M if $[M] \to \mu_x$ under

$$H_n(M) \xrightarrow{\cong} H_n(M, M - \{x\})$$

$$[M] \longmapsto \mu_x$$

see over 13

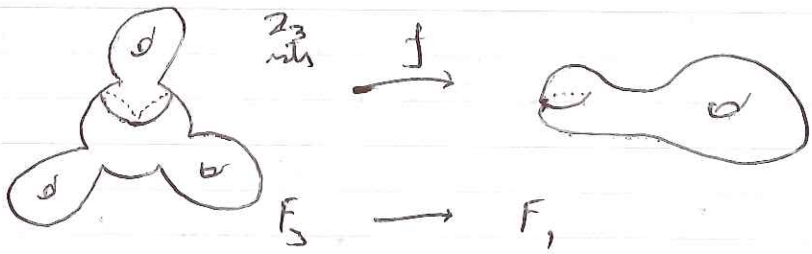
Definition: If $f : M^n \to N^n$ is a map of two connected n-dim. oriented, closed manifolds, then the degree of f

is the number determined by $f_*([M]) = \deg(f)\,[N]$.

Note: this agrees with our previous definitions and can be calculated in the same way.

replace S^n with M, N in Hatcher diagrams. See Thm 3.26

Ex:



$$\deg(f) = 3$$

Analogous to the cup product, we can define the cap product

$$\frown : C_k(X) \times C^l(X) \longrightarrow C_{k-l}(X) \qquad k \geq l$$

$$\left(\sigma : [v_0, \dots, v_k] \to X,\ \varphi\right) \longmapsto \varphi\left(\sigma|_{[v_0, \dots, v_l]}\right) \sigma|_{[v_l, \dots, v_k]}$$

$\partial(\sigma \frown \varphi) = (-1)^l (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$

Checks:

cycle $\frown$ cocycle = cycle
cycle $\frown$ coboundary = boundary
boundary $\frown$ cocycle = boundary

and hence get a map

$$H_k(X) \times H^l(X) \xrightarrow{\frown} H_{k-l}(X)$$

Theorem: Let M be a closed, oriented manifold of dim n. Then

$$D : H^k(M) \longrightarrow H_{n-k}(M)$$
$$\varphi \longmapsto D(\varphi) := [M] \frown \varphi$$

is an isomorphism.

unoriented mflds can have a fundamental class without this

If M is not oriented then

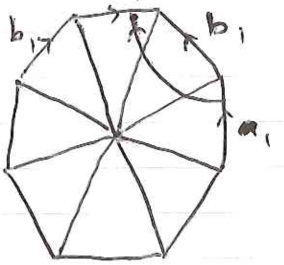
$$D : H^k(M; \mathbb{Z}_2) \longrightarrow H_{n-k}(M, \mathbb{Z}_2)$$
$$\varphi \longmapsto D(\varphi) = [M] \frown \varphi$$

is an isomorphism.

Ex: $M = F_2$ $\quad [M] =$ sum of 2-simplices

$$H_1 = \langle a_1, b_1, a_2, b_2 \rangle$$

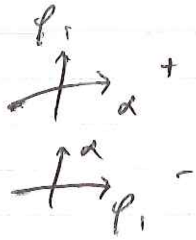
$$H^1 = \langle a_1^*, b_1^*, a_2^*, b_2^* \rangle$$



$$\left.\begin{array}{l} a_1^* : C_1 \to \mathbb{Z} \\ \qquad \alpha \mapsto \pm 1 \\ \qquad\qquad\quad 0 \end{array}\right\} \alpha \cap \varphi_1 = \phi$$

$\varphi_1 \simeq b_1$

$$D(a_1^*) = [M] \cap \varphi^* \quad \dots \quad = b_1$$



HATCHER pg 241

An element of $H_n(M)$ whose image in $H_n(M, M - \{x\})$ is a generator for all $x \in M$ is called a fundamental class for M.

A fundamental class exists iff M is closed and orientable.

see pg 236
HATCHER

POINCARÉ DUALITY

Cap product: $\sigma \frown \varphi = \varphi(\sigma|_{[v_0,\dots,v_\ell]})\, \sigma|_{[v_\ell,\dots,v_k]}$

$\partial(\sigma \frown \varphi) = (-1)^\ell((\partial\sigma \frown \varphi) - (\sigma \frown \partial^*\varphi))$

$[M] \in H_n(M)$ fundamental class (oriented) of a closed n-dim. manifold

$\in H_n(M, \frac{\mathbb{Z}}{2\mathbb{Z}})$ (non-oriented)

$\varphi \in H^\ell(M)$

$[M] \frown \varphi \in H_{n-\ell}(M)$

on chains / cochains

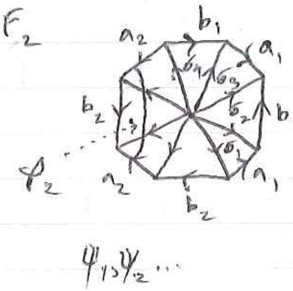
$$\begin{array}{ccc} D: C^k(M) & \longrightarrow & C_{n-k}(M) \\ \downarrow \partial^* & & \downarrow \partial \\ C^{k+1}(M) & \longrightarrow & C_{n-k-1}(M) \end{array}$$

commutes up to a sign, where $D(\varphi) := [M] \frown \varphi$ (representative...)

$(\partial([M] \frown \varphi) = (-1)^{k+1}[M] \frown \partial^*\varphi)$

Th.: POINCARÉ DUALITY

Ex: F_2



$[M] = \sigma_1 + \sigma_2 - \sigma_3 - \sigma_4 + \sigma_5 + \sigma_6 - \sigma_7 - \sigma_8$

$\partial[M] = 0$

defines fundamental class.

$H_1(F_2) = \langle a_1, b_1, a_2, b_2 \rangle$

$H^1(F_2) = \langle a_1^*, b_1^*, a_2^*, b_2^* \rangle$

a_1^* is the cocycle that assigns 1 to each 1-simplex that meets the curve φ_1. Sim a_2^*, b_1^*, b_2^* (in general in this construction, would need ± 1) (convention about crossing)

$Da_1^* = [M] \frown a_1^* = b_1$

Similarly
$$Db_1^* = -a_1$$
$$Da_2^* = b_2$$
$$Db_2^* = -a_2$$

Interpretation of the cup product in terms of intersection product

$$\begin{array}{ccc}
H^k(M) \otimes H^l(M) & \xrightarrow{\ \cup\ } & H^{k+l}(M) \\
D \downarrow \cong \qquad \cong \downarrow D & & \cong \downarrow D \\
H_{n-k}(M) \otimes H_{n-l}(M) & \xrightarrow{\ \pitchfork\ } & H_{n-k-l}(M)
\end{array}$$

intersection product

$$a_1^* \cup b_1^* = [M]^* = -b_1^* \cup a_1^*$$
$$a_2^* \cup b_2^* = [M]^* = -b_2^* \cup a_2^*$$ all other products are zero.

$D(a_1^* \cup b_1^*)$ = intersection product of $Da_1^* = b_1$ and $Db_1^* = -a_1$
$= -b_1 \pitchfork a_1$

Ex: $M = T^3 = S^1 \times S^1 \times S^1$

$$\begin{array}{ccc}
H^1 \otimes H^1 & \xrightarrow{\ \cup\ } & H^2 \\
\cong \downarrow & & \downarrow \cong \\
H_2 \otimes H_2 & \xrightarrow{\ \pitchfork\ } & H_1
\end{array}$$

THINK

$$\begin{array}{ccc}
(S^1 \times 0 \times 0)^* \otimes (0 \times 0 \times S^1)^* & \mapsto & (S^1 \times 0 \times S^1)^* \\
\downarrow & & \downarrow \\
(0 \times S^1 \times S^1) \otimes (S^1 \times S^1 \times 0)^* & \mapsto & 0 \times S^1 \times 0
\end{array}$$

Ex: $M = F_2$ $\quad \underset{0 =}{D(a_1^* \cup a_2^*)} = Da_1^* \pitchfork Da_2^* = b_1 \pitchfork b_2$

Pull loops into transversal intersection

15 TRANSFER MAP

Recall a covering space of X is a space $\tilde{X}$ (not necessarily path-connected) with a surjective map

$$p : \tilde{X} \to X$$

s.t. there exists a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of open sets in X s.t. for all $\alpha \in I$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each mapped homeomorphically onto U_α.

p induces a map in homology $p_* : H_k(\tilde{X}) \to H_k(X)$

Assume that $p : \tilde{X} \to X$ is finitely sheeted, say n-sheeted. (i.e. $|p^{-1}(x)| = n$ for all $x \in X$). Then p induces a "wrong way map"

Consider $\underset{\text{transfer}}{\mathrm{trf}} : C_k^{\mathcal{U}}(X) \longrightarrow C_k(\tilde{X})$

$$\underset{\text{with } \sigma(\Delta^k) \subset U_\alpha}{\sigma : \Delta^k \to X} \longmapsto \sum \tilde{\sigma}$$

$\tilde{\sigma} : \Delta^k \to \tilde{X}$ s.t. $p \circ \tilde{\sigma} = \sigma$.

Note, this is well defined as the sum is finite. Also, note that trf is a chain map, $\mathrm{trf}(\partial\sigma) = \partial\,\mathrm{trf}(\sigma)$. Furthermore

$$p_\# \circ \mathrm{trf}(\sigma) = n \cdot \sigma.$$

Proposition: If $p : \tilde{X} \to X$ is an n-sheeted cover, then

$$(\mathrm{trf})_* : H_k(X) \to H_k(\tilde{X}) \text{ and } p_* \circ (\mathrm{trf})_* = (\times n)$$

Assume now that X is given as a finite Δ-set with small enough simplices s.t. for each simplex σ, $p^{-1}(\sigma)$ is the disjoint union of simplices in $\tilde{X}$.

Proposition: $\chi(\tilde{X}) = n \cdot \chi(X)$

Proof: $\chi(\tilde{X}) = \sum_{k=0} (-1)^k \operatorname{rank} C_k^\Delta(\tilde{X}) = \sum_{k=0} (-1)^k \#\, k\text{-simplices in } \tilde{X}$

$$= \sum_{k=0} (-1)^k n \cdot \#\, k\text{-simplices } X = n\chi(X).$$

Application: The only non-trivial finite group G acting freely on S^{2n} is the group of order 2.

Note: $\mathbb{Z}/2\mathbb{Z}$ acts freely on S^k via $x \mapsto -x$

Proof: Assume S^{2n} has a Δ-structure s.t. G permutes all simplices. Let $X = \frac{S^{2n}}{G}$ be the orbit space and $p: S^{2n} \to X$. By proposition

$$2 = \chi(S^{2n}) = |G|\chi(X). \quad \text{So} \quad |G| = 2.$$

Application: $\frac{\mathbb{Z}}{m\mathbb{Z}}$ acts freely on $S^{2n-1} \subseteq \mathbb{C}^n$ generated by

$$(z_1, \dots, z_n) \longmapsto e^{\frac{2\pi i}{m}}(z_1, \dots, z_n).$$

Let $L_m^{(n)} := \frac{S^{2n-1}}{\mathbb{Z}/m\mathbb{Z}}$ be the orbit space, called a lens space. (note: $L_2^{(n)} = \mathbb{RP}^{2n-1}$).

By proposition we have

$$\begin{tikzcd} H_k(L_m^{(n)}) \arrow[r, "(\mathrm{tr} f)_*"] \arrow[dr, "\times m"'] & H_k(S^{2n-1}) \arrow[d, "p_*"] \\ & H_k(L_m^{(n)}) \end{tikzcd}$$

so $H_k(L_m^{(n)}) = 0$ for $0 < k < 2n-1$

Application: Homology of (finite) groups. Let G be a discrete group, $\tilde{X}$ be a contractible Δ-complex with a free G-action (G acts freely on simplices). The orbit space

$$BG := \frac{\tilde{X}}{G}$$

is unique up to homotopy.

Ex: $G = \mathbb{Z}$ and $\tilde{X} = \mathbb{R}$, then $BG = \frac{\mathbb{R}}{\mathbb{Z}} = S^1$

Ex: $G = \frac{\mathbb{Z}}{m\mathbb{Z}}$ acts freely on $S^\infty = \bigcup_n S^{2n-1}$ and S^∞ is contractible. Thus

$$BG := \frac{S^\infty}{\mathbb{Z}/m\mathbb{Z}} = L_m^{(\infty)}.$$

$$B\frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{RP}^\infty.$$

$$H_k^{EM}(G) := H_k(BG)$$

Proposition: For finite G, $|G|\, H_k^{EM}(G) = 0$ for $k > 0$.

Proof:

$$\begin{tikzcd}
H_k^{EM}(G) \cong H_k(BG) \cong H_k\left(\frac{\tilde{X}}{G}\right) \arrow[r, "(\mathrm{trf})_*"] \arrow[dr, "\times |G|"'] & H_k(\tilde{X}) \arrow[d, "p_*"] \\
& H_k(BG) \cong H_k^{EM}(G)
\end{tikzcd}$$

By proposition, the composition is multiplication by $|G|$.
But $H_k(\tilde{X}) = 0$ for all $k > 0$.