Lecture 18 - Hopf Invariant One

Thm 1 If $n \neq 1,2,4,8$, there are no maps of Hopf invariant one $S^n \to S^0$.

To parse this we need some definitions. This is a stable formulation. More original:

Consider a map $S^{n+k} \xrightarrow{f} S^n$, and pick 2 regular values of the map. The preimages of these points are manifolds, and we can take their intersection. If this yields a number (as a collection of oriented pts) then this is the Hopf invariant. Since $\dim f^{-1}(p) = k$, we must have $2k+1 = n+k$ for these to link in a 0-manifold.

Intersection pairing $\longleftrightarrow$ cup product. So consider $C(f)$.

Def Let $f: S^{2n-1} \to S^n$. Let $x_n \in H^n(C(f))$ & $x_{2n} \in H^{2n}(C(f))$ be generators which restrict to / are induced by the orientation class of the spheres. The Hopf invariant of f is defined by

$$x_n^2 = H(f)\, x_{2n}.$$

The three classical examples:

① $$\begin{array}{ccc} S^3 & \xrightarrow{\eta} & S^2 \\ \| & & \| \\ S(\mathbb{C}^2) & \longrightarrow & \mathbb{C}^+ \end{array} \qquad C(\eta) = \mathbb{C}P^2, \quad H^*\mathbb{C}P^2 = \mathbb{Z}[x_2]/x_2^3$$

$$(v,w) \mapsto \begin{cases} v/w & w \neq 0 \\ + & w = 0 \end{cases}$$

② $$\begin{array}{ccc} S^7 & \xrightarrow{\nu} & S^4 \\ \| & & \| \\ S(\mathbb{H}^2) & \longrightarrow & \mathbb{H}^+ \end{array} \qquad C(\nu) = \mathbb{H}P^2, \quad H^*\mathbb{H}P^2 = \mathbb{Z}[x_4]/x_4^3$$

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③ $$\begin{array}{ccc} S^{15} & \xrightarrow{\sigma} & S^8 \\ \| & & \| \\ S(\mathbb{O}^2) & \longrightarrow & \mathbb{O}^+ \end{array} \qquad C(\sigma) = \mathbb{O}P^2, \quad H^*\mathbb{O}P^2 = \mathbb{Z}[x_8]/x_8^3$$

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These examples all arise from a division algebra structure on $\mathbb{R}^n$.

Thm 2 If $\mathbb{R}^n$ has the structure of a division alg, then $\exists\ S^{2n-1} \to S^n$ of Hopf inv. 1.

The map is the one just as in the classical cases.

<u>Cor 1</u> The only division algebras on $\mathbb{R}^n$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, & $\mathbb{O}$.

<u>Fact</u> The Hopf invariant is additive, and we can always find maps of Hopf inv. 2.

So Hopf inv 1 $\leftrightarrow$ Hopf inv. odd.

This also lets us stabilize: $Sq^n x_n = x_n^2 = H(f) \cdot x_{2n}$.

Sq^n commutes w/ Σ, and spheres suspend to spheres.

<u>Def</u> If $f: S^{n-1} \longrightarrow S^0$, define the mod 2 Hopf invariant of f by $Sq^n x_0 = H(f) x_n$ in $H^* C(f)$.

<u>Cor 2</u> If n is not a power of 2, then $H(f) = 0$.

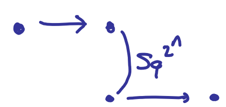
<u>Proof</u>: Consider $H^* C(f)$:

$$\begin{array}{ccc} n & \bullet & \mathbb{F}_2 \\ & \big\uparrow\, Sq^n = H(f)\cdot & \\ 0 & \bullet & \mathbb{F}_2 \end{array}$$

If n is not a power of 2, then the Adem rels show that $Sq^n = \sum_{i=1}^{n-1} a_i Sq^i Sq^{n-i}$. Since $H^{n-i}(C(f)) = 0$ for $0 < i < n$, $Sq^n x_0 = 0$. ☒

Pause here to recognize the connection w/ the Adams SS.

$$\left\{ \begin{array}{c} \text{Map of Hopf Inv 1} \\ S^{2^n-1} \longrightarrow S^0 \\ \text{exists} \end{array} \right\} \longleftrightarrow \left\{ H^*(C(f)) \text{ sits as} \right\} \longleftrightarrow \left\{ \begin{array}{c} h_n \text{ survives} \\ \text{the Adams} \\ \text{S.S.} \end{array} \right\}$$



Saw in <u>Lecture 17</u> that h_4 does not survive.

<u>Cor 3</u> ① There is no map of Hopf inv. 1 $S^{31} \to S^{16}$

② $\mathbb{R}^{16}$ is not a division alg.

Building on <u>Cor 2</u>, Adams solved this in general.

<u>Thm 3</u> There is a d_2-differential: $d_2(h_n) = h_0 h_{n-1}^2$, $n \geq 4$.

Adams showed this by analyzing the Adem relation built out of

$Sq^{2^{n-1}} \cdot Sq^{2^{n-1}}$. This gives us a secondary operation (relations always do). Adams showed more over that there is an A-linear comb of secondary ops ϕ s.t.

$$Sq^{2^n}(a) \in \phi(a) \quad \text{for all } a, \quad n \geq 4.$$

This decomposition is equivalent to the Adams differential.

Here is an outline of another argument: Steenrod ops in Ext.

Def Let A denote the associative algebra gen by Sq^n for $n \geq 0$ subject to all the usual relations *except* $Sq^0 = Id$. This is called the big Steenrod alg / alg of power ops.

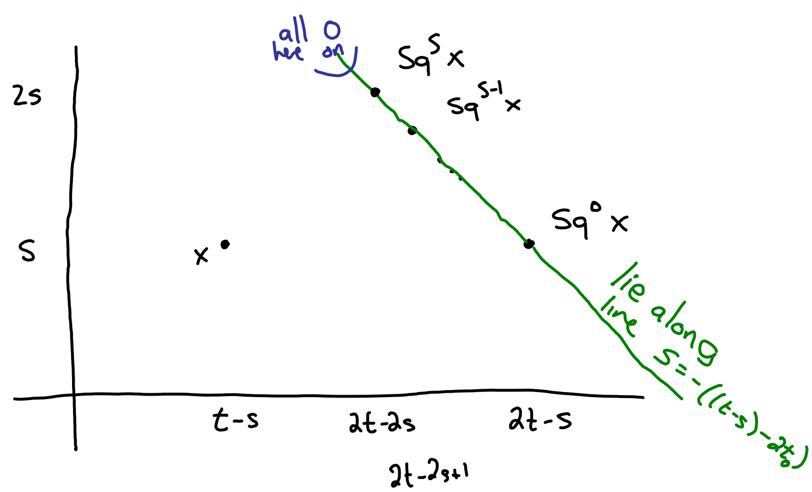
Thm 4 $Ext_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ is an unstable module over A:

1. $Sq^n : Ext^{s,t} \longrightarrow Ext^{s+n, 2t}$
2. $Sq^s : Ext^{s,t} \longrightarrow Ext^{2s, 2t}$ is the square
3. $Sq^{>s} : Ext^{s,t} \longrightarrow Ext^{>2s, 2t}$ is zero.
4. Sq^0 is the lift of Frobenius in cobar

$$[x_0 | \cdots | x_n] \xmapsto{Sq^0} [x_0^2 | \cdots | x_n^2] .$$

This was shown in great generality by May.

Picture of Steenrod ops:



These tie in with the Adams structure: extended powers:

$$D_2 = E\mathbb{Z}/2_+ \underset{\mathbb{Z}/2}{\wedge} (X \wedge X).$$

Can also look at "skeleta" $L_n = S^n_+ \underset{\mathbb{Z}/2}{\wedge} (X \wedge X)$.

Moral Theorem The skeletal filtration of $D_2(S^0)$ maps to the Adams filt of S^0.

$$f: S^n \to S^0 \quad \Rightarrow \quad D_2(f): D_2(S^n) \longrightarrow D_2(S^0) \longrightarrow S^0$$

looks like f^2 & higher stuff

So we see f^2 in the right filt, the $2n+1$ cell of $D_2 S^n$ as $Sq^{n-1} f$, etc.

In this framework:

$$\left\{ \begin{array}{c} \text{attaching maps} \\ \text{in } D_2(S^n) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Adams} \\ \text{differentials} \end{array} \right\}$$

(cells are null-homotopies of others)

Thm 5 If $i \equiv t \ (2)$, & $x \in Ext^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$, then

$$d_2(Sq^i(x)) = h_0 \cdot Sq^{i+1}(x).$$

this is the basic attaching map in $D_2 S^n$. the next is η & we see all of $\operatorname{Im} J$.

Now $h_i \in Ext^{1,2^i}$ is represented in the cobar complex by $[\xi_1^{2^i}]$ $(\longleftrightarrow Sq^{2^i})$. So

$$Sq^*(h_i) = h_{i+1} + h_i^2.$$

$$h_{i+1} = [\xi_1^{2^{i+1}}] = [(\xi_1^{2^i})^2] = Sq^0 h_i, \qquad h_i^2 = Sq^1 h_i$$

Cor 4 $d_2(h_{i+1}) = h_0 h_i^2$ for all $i > 0$.

Pf: Apply Thm 5 to h_i: $d_2(Sq^0(h_i)) = h_0 Sq^1(h_i) = h_0 h_i^2$. ☒

No prop w/ $i < 3$, since $h_0 h_i^2 = 0$ on E_2.