Lecture 12 - Bockstein Spectral Sequences

Cover a very general kind of spectral sequence: Bockstein SS.

There are two main flavors which differ b bookkeeping conventions

Let R be a ring (normally $\mathbb{Z}$) and let $p \in R$ be a central element.

Let $M_\bullet$ be a dg R-mod.

Def The Bockstein filtration of $M_\bullet$ with respect to p is given by

$$F_{-n}M_\bullet = p^n M_\bullet$$

Note that if $m \in M$ is such that mult by p on m is invertible (m is then infinitely p-divisible), then $m \in F_nM$ for all n.

Prop 1 ① $F_{-\infty}M = \varprojlim F_nM = \bigcap_{n \geq 0} F_nM = \{ m \in M \mid \forall n, \exists m' \text{ s.t. } p^n \cdot m' = m \}$.

② $F_\infty M = \varinjlim F_nM = M$

The filtrations we've encountered before are not like this. There F_∞ was zero & $F_{-\infty} = M$.

Def A filtration is exhaustive if $M = \varinjlim F_nM$

" " " Hausdorff if $\{0\} = \varprojlim F_nM$.

These notions are important since they tell us ① how much the filtration sees and ② if it is fine enough to separate points.

Cor 1 If M has no infinitely p-divisible elements, then the Bockstein filt is Hausdorff.

Over $\mathbb{Z}$ this is usuall obvious.

Now consider the short exact sequences

(1) $$0 \to F_{n-1}M \to F_nM \to F_nM/F_{n-1}M \to 0$$

Essentially by construction, $F_nM/F_{n-1}M$ is d.g. R/p-module. This alone makes it easier to work with in good cases.

Cor 2 If $m \in F_nM$ has $p^{n+1} \mid d(m)$, then in $F_nM/F_{n-1}M$, m is a cycle.

Applying homology turns the SES (i) into a LES:

$$\begin{tikzcd} H_{n+m}(F_{n-1}) \arrow[r] & H_{n+m}(F_n) \arrow[d] \\ H_{n+m-1}(F_{n-1}) & H_{n+m}(F_n/F_{n-1}) \arrow[l] \end{tikzcd}$$

And we get an exact couple: $D^{p,q} = H_{p+q}(F_p)$

$E^{p,q} = H_{p+q}(F_p/F_{p-1})$

$$\left.\begin{array}{ll} i: & D^{p-1,q+1} \longrightarrow D^{p,q} \\ & \quad \| \qquad\qquad\quad \| \\ & H_{p+q}(F_{p-1}) \to H_{p+q}(F_p) \\ j: & D^{p,q} \longrightarrow E^{p,q} \\ & \quad \| \qquad\qquad\quad \| \\ & H_{p+q}(F_p) \to H_{p+q}(F_p/F_{p-1}) \\ k: & E^{p,q} \longrightarrow D^{p-1,q} \\ & \quad \| \qquad\qquad\quad \| \\ & H_{p+q}(F_p/F_{p-1}) \to H_{p+q-1}(F_{p-1}) \end{array}\right\} \text{ The usual maps}$$

The associated SS is the Bockstein spectral sequence.

The differentials in the Bockstein SS measure the p divisibility of differentials and hence detect p-torsion.

<u>Example 1</u> $M_\bullet = 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{0}{\mathbb{Z}} \to 0$; $p = 2.$

This is the key to the other approach

Then $F_{-n}M_\bullet = \begin{array}{c} 2^n \cdot \mathbb{Z} \\ 2\cdot\downarrow \\ 2^n \cdot \mathbb{Z} \end{array} \cong \begin{array}{c} \mathbb{Z} \\ \downarrow \cdot 2 \\ \mathbb{Z} \end{array}$; $F_{-n}M/F_{-n-1}M = \begin{array}{c} 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \\ 0\downarrow \\ 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \end{array} \cong \begin{array}{c} \mathbb{Z}/2\mathbb{Z} \\ \downarrow \cdot 0 \\ \mathbb{Z}/2 \end{array}$.

Thus $H_*(F_{-n}M/F_{-n-1}M) = \begin{array}{cc} \vdots & \\ 0 & \\ \mathbb{Z}/2 & 1 \\ \mathbb{Z}/2 & 0 \\ 0 & \vdots \\ \vdots & \end{array}$

So $E_1^{p,q} = \begin{cases} \mathbb{Z}/2 & p \leq 0, \quad 0 \leq p+q \leq 1 \\ 0 & \text{o.w.} \end{cases}$

Let's compute d_1: $j \circ k : H_{p+q}(F_p/F_{p-1}) \to H_{p+q-1}(F_{p-1}/F_{p-2})$.

(for degree reasons, this only matters if $p+q = 1$)

k = connecting hom: We chase through the diagram to see the generator of $H_1(F_{-n}/F_{-n-1})$ maps to the gen of $H_0(F_{-n-1})$.

$$\begin{tikzcd}
 & & 2^n \arrow[r, mapsto] \arrow[ddd, bend right] & 2^n \arrow[d, phantom, "\Uparrow" description] & \\
0 \arrow[r] & 2^{n+1}\mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 2^n\mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & 2^{n+1}\mathbb{Z} \arrow[r] & 2^n\mathbb{Z} \arrow[r] & 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \arrow[r] & 0 \\
 & 2^{n+1} \arrow[r, mapsto] & 2^{n+1} \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

The map $H_0(F_{-n-1}) \to H_0(F_{-n-1}/F_{-n-2})$ is the obvious surjection, so we conclude that

$$d_1 : H_1(F_{-n}/F_{-n-1}) \to H_0(F_{-n-1}/F_{-n-2}) \quad \text{is an iso.}$$

$$\Rightarrow \quad E_2^{p,q} = \begin{cases} \mathbb{Z}/2 & p=q=0 \\ 0 & \text{o.w.} \end{cases}$$

Now note also that $H_*\left(\begin{smallmatrix}\mathbb{Z}\\ \downarrow \cdot 2\\ \mathbb{Z}\end{smallmatrix}\right) = \begin{cases} \mathbb{Z}/2 & *=0 \\ 0 & \text{o.w} \end{cases}$, as expected.

Example 2 Same set-up, but $\mathbb{Z} \xrightarrow[\cdot 4]{} \mathbb{Z}$.

Again, $E_1^{p,q} = \begin{cases} \mathbb{Z}/2 & p \leq 0, \ p+q=0,1 \\ 0 & \text{o.w.} \end{cases}$

d_1:

$$\begin{tikzcd}
 & & 2^n \arrow[r, mapsto] \arrow[ddd, mapsto, bend right] & 2^n \arrow[d, phantom, "\Uparrow" description] \arrow[ddd, bend left] & \\
0 \arrow[r] & 2^{n+1}\mathbb{Z} \arrow[r] \arrow[d, "\cdot 4"] & 2^n\mathbb{Z} \arrow[r] \arrow[d, "\cdot 4"] & 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & 2^{n+1}\mathbb{Z} \arrow[r] & 2^n\mathbb{Z} \arrow[r] & 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \arrow[r] & 0 \\
 & 2^{n+2} \arrow[r, mapsto] & 2^{n+2} \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

But $2^{n+2} \longmapsto 0$ under $F_{-n-1} \to F_{-n-1}/F_{-n-2}$.

$\Rightarrow d_1 = 0$.

For d_2, we compute $j \circ i^{-1} \circ k$. This amounts to extending our diagram:

$$\begin{tikzcd}
 & & & 2^n \arrow[r, mapsto] \arrow[ddd, mapsto, bend right] & 2^n \arrow[d, phantom, "\Uparrow" description] \arrow[ddd, mapsto, bend left] & \\
0 \arrow[r] & 2^{n+2}\mathbb{Z} \arrow[r] \arrow[d, "\cdot 4"] & 2^{n+1}\mathbb{Z} \arrow[r] \arrow[d, "\cdot 4"] & 2^n\mathbb{Z} \arrow[r] \arrow[d, "\cdot 4"] & 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & 2^{n+2}\mathbb{Z} \arrow[r] \arrow[d, phantom, "\Downarrow" description] & 2^{n+1}\mathbb{Z} \arrow[r] & 2^n\mathbb{Z} \arrow[r] & 2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \arrow[r] & 0 \\
 & 2^{n+2} \arrow[r, mapsto] & 2^{n+2} \arrow[r, mapsto] & 2^{n+2} \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

Then reduction mod 2^{n+3} sends 2^{n+2} to the generator $\Rightarrow$

$E_3 = \begin{cases} \mathbb{Z}/2 & p+q=0, \ p=0,-1 \\ 0 & \end{cases}$. Now $H_*\left(\begin{smallmatrix}\mathbb{Z}\\ \downarrow \cdot 4\\ \mathbb{Z}\end{smallmatrix}\right) = \begin{cases} \mathbb{Z}/4 & *=0, \\ 0 & \text{o.w.} \end{cases}$

Prop 2 To compute a d_r-differential, lift to an integral class, apply the differential, reduce mod $r+1$ filtrations lower.

Thm 1 The Bockstein SS converges to $H_*(M/\text{p-divisible elemts})$.

With care, can detect $\cdot 2$.

The second version is slightly less cumbersome: Here we use the isom $2^n\mathbb{Z} \cong \mathbb{Z}$ & $2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \cong \mathbb{Z}/2$.

This allows us to have just one copy:

$$0 \to M_\bullet \xrightarrow{\cdot 2} M_\bullet \to M_\bullet/2 \to 0$$

gives a *singly graded* exact couple $D^p = H_p(M)$

$E^p = H_p(M/2)$

Prop 3 To compute a d_r-differential, lift to an integral class, apply d, divide by 2^r, and reduce mod 2.

Thus a d_r-diff detects 2^r-torsion in $H_*(M)$

Thm 2 This Bockstein SS converges to $H_*(M/\text{2-divisibles})/\text{torsion}$.

This version obviously generalizes to handle the case M a free R-module, $p \in R$ not a zero-divisor (so $0 \to M \xrightarrow{\cdot p} M \to M/p \to 0$ is exact).