Homotopy Groups & Postnikov Section

Note Title 2/19/2009

Can use the Serre spectral sequence to understand homotopy gps.

Def If $F \to E \xrightarrow{\pi} B$ is a fibration & $f: X \to B$ is a map, define the <u>induced fibration</u> f^*E over X by

$$f^*(E) = \{(x,e) \mid f(x) = \pi(e)\} \subseteq X \times E.$$

The projection $X \times E \to X$ gives rise to a map $f^*(E) \to X$

Prop 1 a) $f^*(E) \to X$ is a fibration w/ fiber F &

b)
$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & f^*(E) \arrow[r] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ F \arrow[r] & E \arrow[r, "\pi"'] & B \end{tikzcd}$$
is a map of fibrations

<u>Prop 1 b)</u> implies that we get a map of Serre SSs:

$$\begin{tikzcd} H^p(B; H^q(F)) \arrow[r, Rightarrow] \arrow[d, "f^*"] & H^{p+q}(E) \arrow[d, "\tilde{f}^*"] \\ H^p(X; H^q(F)) \arrow[r, Rightarrow] & H^{p+q}(f^*E) \end{tikzcd}$$

We will use this <u>a</u> <u>lot</u> to compute.

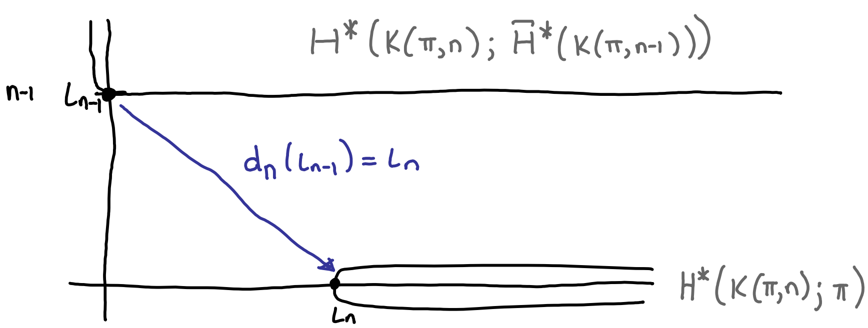
Two big examples:

① $B = K(\pi, n)$, $E = PK(\pi,n)$, $F = K(\pi, n-1)$

& $f: X \to B \longleftrightarrow f^*\iota_n \in H^n(B; \pi)$

<u>Prop 2</u> In the SSS for $f^*PK(\pi,n)$, $\iota_{n-1} \in H^0(B; H^{n-1}(F;\pi))$ transgresses to $f^*\iota_n \in H^n(B)$.

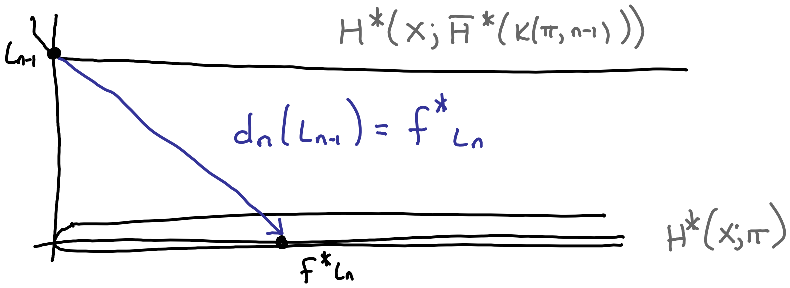
The SSS for $F \to E \to B$ looks like



This is the key result: naturality!

Now we use naturality: on $E_2^{p,0}$, the map is f^* & on $E_2^{0,q}$, it's an iso

this gives us the relevant portion of the SSS for $f^*PK(\pi,n)$.



$\square$

The upshot is that $\pi^*(f^*\iota_n) = 0$ in $H^n(f^*PK(\pi,n))$ & the fundamental class of the fiber will not survive the Serre SS.

② Associated to the SES $\mathbb{Z}/2 \to \mathbb{Z}/2^k \to \mathbb{Z}/2^{k-1}$

are fibrations
$$\begin{array}{ccc} K(\mathbb{Z}/2,n) \to & K(\mathbb{Z}/2^k,n) \\ & \downarrow \\ & K(\mathbb{Z}/2^{k-1},n) \end{array},$$

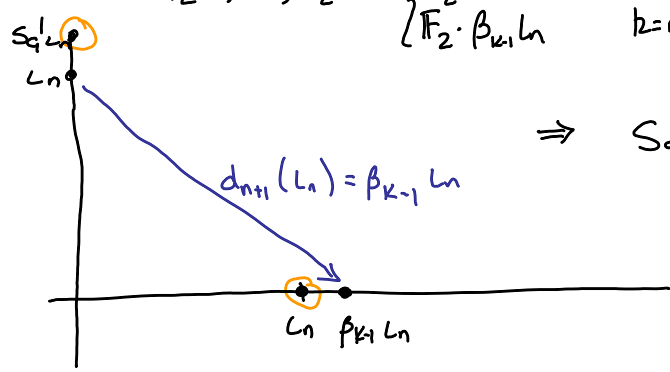
classified by a map $K(\mathbb{Z}/2^{k-1},n) \xrightarrow{\beta_{k-1}} K(\mathbb{Z}/2,n+1)$.

We can use this to inductively define the higher Bocksteins β_k.

Prop 3 $\beta_k(x) = Sq^1(\text{class that kills } \beta_{k-1}(x))$.

Pf: We'll show the universal example:

$$H^k(K(\mathbb{Z}/2^{k-1},n);\mathbb{F}_2) = \begin{cases} \mathbb{F}_2\cdot\iota_n & k=n \\ \mathbb{F}_2\cdot\beta_{k-1}\iota_n & k=n+1 \end{cases}, \text{ so in the SSS, have}$$



$\Rightarrow Sq^1\iota_n$ generates $H^{n+1}(K(\mathbb{Z}/2^k,n);\mathbb{F}_2)$

$\Rightarrow Sq^1\iota_n = \beta_k\iota_n$

$\square$

Both of these examples will be essential.

Remark This is the same idea as a Massey product: $Sq^1\beta_k = 0$,

so if $\beta_k(x) = 0$, can form $\langle Sq^1, \beta_k, x\rangle \ni \beta_{k+1}(x)$.

II. Postnikov Tower

Def If X is s.c. the Postnikov tower of X is a tower of fibrations $F_i \to X_i = P_i(X)$, together with compatible
$$\downarrow$$
$$X_{i-1}$$
maps $X \xrightarrow{f_i} X_i$ s.t. $\pi_k(X_i) = \begin{cases} \pi_k(X) & k \le i \\ 0 & k > i \end{cases}$

& the isom is induced by f_i.

Under this set-up, $F_i = K(\pi_i(X), i)$ & can assume $X_i \to X_{i-1}$ is a principal F_i-bundle. $\Rightarrow$ $X_i \to X_{i-1}$ is classified by a map $X_{i-1} \longrightarrow K(\pi_i, i+1)$.

Def The k-invariants of X are the composites $K(\pi_{i-1}, i-1) \to K(\pi_i, i+1)$.

In practice, we compute these by knowing $H^*(X)$ and $H^*(X_n)$ for all n.

This is actually a kind of localization (we localize w.r.t $\{S^{\ge n} \to *\}$) $\Rightarrow$ functorial.

Ex 1 $\mathbb{Q}$ homotopy gps of spheres.

Thm 1
$$\pi_* S^{2n-1}_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & * = 2n-1 \\ 0 & * \ne 2n-1 \end{cases}$$
$$\pi_* S^{2n}_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & * = 2n, 4n-1 \\ 0 & * \ne 2n, 4n-1 \end{cases}$$

Pf Hurewicz Theorem $\Rightarrow$ $S^n \to K(\mathbb{Z}, n)$ is the n^{th} Postnikov section of S^n. Also know that $S^n \to P_m(S^n)$ induces an isom
$$H^k(P_m(S^n)) \to H^k(S^n) \quad \text{for all } k \le m.$$

If n is odd, then $H^*(K(\mathbb{Q}, n)) \cong E(\iota_n) \cong H^*(S^n)$

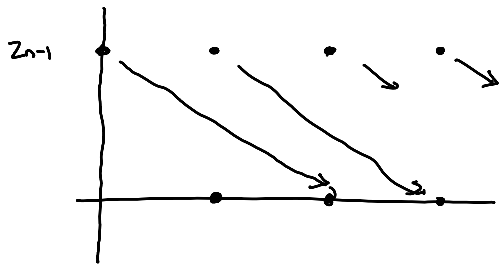
& the map $S^n_{\mathbb{Q}} \to K(\mathbb{Q}, n)$ induces this isom. Whitehead thm $\Rightarrow$ $S^n_{\mathbb{Q}} \xrightarrow{\simeq} K(\mathbb{Q}, n)$.

If n is even, then $H^*(K(\mathbb{Q}, n)) = \mathbb{Q}[\iota_n] \longrightarrow E(\iota_n) = H^*(S^n)$.

To make the cohom match up, we must kill ι_n^2.

Let P be the pull-back of $P(K(\mathbb{Q}, 2n))$ along $\iota_n^2 : K(\mathbb{Q}, n) \to K(\mathbb{Q}, 2n)$.

By Prop 2, the SSS for $H^*(P)$ is



$\Rightarrow \quad H^*(P) = E(\iota_n)$.

Since $\iota_n^2 \longmapsto 0 \in H^*(S^n)$, $S^n \longrightarrow K(\mathbb{Q}, n)$ lifts to P.

Then by naturality, the induced map $H^*(P) \to H^*(S^n)$ is an iso. $\Rightarrow$ $S^n_{\mathbb{Q}} \simeq P$. ($P$ is the Postnikov tower of S^n, sped up). ◻

<u>Ex 2</u> Stable homotopy gps of spheres: $\pi_k^S S^0 = \pi_{k+n} S^n$, $n \gg 0$.

<u>Thm 2</u>
1) $\pi_0^S = \mathbb{Z}$
2) $\pi_1^S = \mathbb{Z}/2$
3) $\pi_2^S = \mathbb{Z}/2$
4) $\pi_3^S = \mathbb{Z}/24$

<u>Pf</u> 1) is classical: it's the Hurewicz thm. We'll work at $p=2$. $p>2$ is hw.
Start with $S^n \longrightarrow K(\mathbb{Z}, n)$ & compare $H^*(-; \mathbb{Z}/2)$. If $n \gg 0$, then $n+5 \ll 2n$, so we see no products in $H^*(-; \mathbb{Z}/2)$.

dim	S^n	$K(\mathbb{Z},n)$
n	ι_n	ι_n
$n+1$	—	—
$n+2$	—	$Sq^2 \iota_n$
$n+3$	—	$Sq^3 \iota_n$
$n+4$	—	$Sq^4 \iota_n$
$n+5$	—	$Sq^5 \iota_n$

Must eventually kill these off.

We see that the only way to get rid of $Sq^2 \iota_n$ is to have it as a k-invariant:

$$\begin{array}{ccc} K(\mathbb{Z}/2, n+1) \to P_{n+1}(S^n) & & \\ \downarrow & & \\ K(\mathbb{Z}, n) & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2) \end{array}$$

Since we are "stable", the SSS looks like a LES from transgressive elements (the only things are on $E_2^{0,*}$ & $E_2^{*,0}$)

$K(\mathbb{Z}, n)$	$K(\mathbb{Z}/2, n+1)$
ι_n	
$-$	
	ι_{n+1}
$Sq^2 \iota_n$	$Sq^1 \iota_{n+1}$
$Sq^3 \iota_n$	$Sq^2 \iota_{n+1}$
$Sq^4 \iota_n$	$Sq^3 \iota_{n+1}$, $Sq^2 Sq^1 \iota_{n+1}$
$Sq^5 \iota_n$	$Sq^4 \iota_{n+1}$, $Sq^3 Sq^1 \iota_{n+1} = \beta_2 Sq^4 \iota_n$ by prop 3

Prop 1

Adem rels

So $H^* P_{n+1} S^n$:

ι_n

$-$

$-$

$Sq^2 \iota_{n+1}$ ← Must kill off $\Rightarrow$

$$\begin{array}{c} P_{n+2} S^n \\ \downarrow \\ P_{n+1} S^n \xrightarrow{Sq^2 \iota_{n+1}} K(\mathbb{Z}/2, 3) \end{array}$$

see Sq^2 & $Sq^1(-)$

$Sq^3 \iota_{n+1}$, $Sq^4 \iota_n$

$Sq^3 Sq^1 \iota_{n+1}$

$P_{n+1} S^n$:	$K(\mathbb{Z}/2, 2)$
ι_n	
$-$	
$-$	ι_{n+2}
$Sq^2 \iota_{n+1}$	$Sq^1 \iota_{n+2}$
$Sq^3 \iota_{n+1}$, $Sq^4 \iota_n$	$Sq^2 \iota_{n+2}$
$Sq^3 Sq^1 \iota_{n+1}$	$Sq^3 \iota_{n+2} = \beta_3 Sq^4 \iota_n$ by Prop 3

Prop 1

Adem

The next class to kill supports β_3, so it is an 8-torsion class, and the next map is $P_{n+2} S^n \xrightarrow{Sq^4 \iota_n} K(\mathbb{Z}/8, n+4)$. $\boxtimes$