Massey Products, the Steenrod Alg, & Kudo Transgression

We'll recast the Kudo transgression thm as a consequence of a lemma for higher products.

**Definition:** Let \((M_*, d)\) be an associative dga, and let \(a \in M_m\),
\(b \in M_n\), \(c \in M_k\) be cycles s.t. \([a \cdot b] = [b \cdot c] = 0\) in \(H^*(M)\).

Then the **Massey product** of \(a, b, c\) is
\[
\langle a, b, c \rangle = \{ [e \cdot c - \bar{a} \cdot f] \mid e \xrightarrow{d} a \cdot b \quad \bar{a} = (-1)^{|a|} a \} \subseteq H^{m+n+k-1}(M).
\]

There is a lot packed into the statement.

1. \(\langle a, b, c \rangle\) is a subset of \(H^{n+m+k-1}(M)\), not an element.
   More on this later.
2. The elements \(e \cdot c - \bar{a} \cdot f\) are cycles.
   This is easy to check:
   \[
d(e \cdot c - \bar{a} \cdot f) = d(e \cdot c) - d(\bar{a} \cdot f) =
   \]
   \[
   (d(e) \cdot c + (-1)^{|e|} e \cdot d(c)) - (d(\bar{a}) \cdot f + (-1)^{|a|} \bar{a} \cdot d(f))
   \]
   \[
   = (a \cdot b) \cdot c - a \cdot (b \cdot c) = 0.
   \]

Often indicate our choice of "null-homotopies" by placing them over what they bound:
\[
e \xrightarrow{f} \langle a, b, c \rangle.
\]

When thusly adorned, we are referring to an element of \(\langle a, b, c \rangle\).

**Prop 1:** \(\langle a, b, c \rangle\) is a coset of \([a] \cdot H^{n+k-1}(M) + H^{m+n-1}(M) \cdot [c]\).

**PF:** We need to show \(\langle a, b, c \rangle - \langle a, b, c \rangle \in a \cdot H^{n+k-1} + H^{m+n-1} \cdot c\).
Since \(e, e' \xrightarrow{d} a \cdot b \quad f, f' \xrightarrow{d} b \cdot c\), \(e - e' \xrightarrow{f} f - f'\) are cycles in \(M_{n+m-1} \uparrow M_{n+k-1}\) resp. Thus by def, we have
\[
\langle a, b, c \rangle - \langle a', b', c' \rangle = (e \cdot c - \bar{a} \cdot f) - (e' \cdot c - \bar{a} \cdot f') \\
= \langle e - e' \rangle \cdot c - \bar{a} \cdot (f - f') \\
\in Z(M_{n \times n-1}) \cdot c + a \cdot Z(M_{n \times n-1}).
\]

Def: The group \( a \cdot H^{n+1} + H^{n+1} \cdot c \) is the indeterminacy of \( \langle a, b, c \rangle \).

If the indeterminacy is zero, we identify \( \langle a, b, c \rangle \), its coset, with the unique element therein.

Ex: Let \( M = E(a, b, e) \), \( d(e) = ab \). Then can depict \( M \) by

\[
\begin{array}{c}
3 \\
2 \\
1 \\
0 \\
\end{array}
\begin{array}{c}
acb \\
ae \(ab eb) \Rightarrow (\cdots) \cdot b \\
\end{array}
\begin{array}{c}
3 \\
2 \\
1 \\
0 \\
\end{array}
\begin{array}{c}
acb \\
ae \eb \\
\end{array}
\begin{array}{c}
3 \\
2 \\
1 \\
0 \\
\end{array}
\begin{array}{c}
This is \\
an again a \\
Poncare duality \\
alg.
\end{array}
\]

Then \( ae = \langle a, a, b \rangle \) while \( eb = \langle a, b, b \rangle \).

1st \( a H^1 + H^1 b = 0 \) (no red or blue lines off 1 line) \( \Rightarrow \) indeterminacy = 0.

Then \( b^2 = a^2 = 0 \) in \( M \), so \( 0 \) is a good null-homotopy. \( d(e) = ab \), so \( e \) is a good null-homotopy:

\[
\langle a, a, b \rangle = -\bar{a} \cdot e = ae \\
\langle a, b, b \rangle = e \cdot b.
\]

Thus, for, this is basically naming.

Prop 2 If \( [a][b] = [b][c] = [c][d] = 0 \), then

\( a \langle b, c, d \rangle = \langle a, b, c \rangle d \)

as subsets of \( H_k(M) \).

This will let us deduce several results about multiplicative relations on later pages of spectral sequences.

In fact, this is a very strong result, since we have control over indeterminacy too!
If $\alpha = (a, b, c)$ and $\langle b, c, d \rangle$, then can form larger products:

$$\langle a, b, c, d \rangle = \text{hd} \cdot \text{eg} \cdot \text{ai}.$$  We can obviously combine this, but the indeterminacy becomes a nightmare.

If $a \in M_\ast$ has $[a] \cdot [a] = 0$, then can consider **Massey Powers**

$$\alpha^{(n)} = (a, \ldots, a).$$

Since each slot is $a$ itself, we can symmetrically choose null-homotopies, making the set of choices slightly smaller,

Can now tie all of this to the Steenrod algebra:

**New unstable axiom:**

- If $|x| = 2i$, then $P^i(x) = x^g$ ← old one
- If $|x| = 2i + 1$, then $\beta P^i(x) \in x^{<g>} ←$ new part

In practice, $x^{<g>}$, when chosen symmetrically, will be small enough to identify it with $\beta P^i(x)$.

**Lemma 1** In the Serre SS, working over a field of char $p$, if $x$ is a polynomial gen (possibly truncated $> p-1$), and if

$$d(x) = y,$$

then $d(yx^{<g>}) = y^{<g>}$.  

As always, the equality is an inclusion if there is indeterminacy.

**Ex:** Universal example:  

$$E_2 = \mathbb{F}_p \left[ X / x^p \right] \otimes E(y) \otimes \mathbb{F}_p \left[ y^{<g>} \right]$$

$$\text{deg } (x, 2n) \quad \text{deg } (2n+1, 0) \quad \text{deg } (2n+2, 0)$$

$p = 5$

This has $d_{2n+1} (x) = y$ and

$$d_{2n(p-1)+1} (x^{<g>}) = y^{<p>}.$$
Combining this with the result about Steenrod ops gives the "Kudo Transgression thm":

**Thm 1** If \( x \) is a transgressive element hitting \( y \), then \( x^p \cdot y \) hits \( \beta^p \cdot y \).

Can use this w/ Borel's thm to get the cohom of odd primary EM spaces.

**Def** Let \( \text{I} = (\varepsilon_0, s_1, \varepsilon_1, s_2, \ldots, s_m, \varepsilon_m) \) be a seq of ints w/ \( \varepsilon_i = 0, 1 \).

\( \text{I} \) is admissible if \( s_i = p \cdot s_{i+1} + \varepsilon_i \). The excess is \( \varepsilon_0 + \sum (2s_i - 2ps_{i+1} - \varepsilon_i) \).

The \( \varepsilon_0 \) part is a little odd, and it makes what follows messier.

**Thm 2 (Cartan)** For \( p \) an odd prime, \( H^*(K(\mathbb{Z}/p, n); \mathbb{Z}_p) \)

\[ = \text{free commutative alg on classes } \mathbb{D}^*(\text{I}) \text{ for } e(\text{I}) < n \]

(or \( e(\text{I}) = n \) on an odd class).

Why the other part? We really put in all classes and their \( p \)-th powers of any form. The last classes are \( p \)-th Massey powers, given by something of excess \( n \), so we need to include them at this stage.