**Constructing the Squares**

We'll give a geometric construction. An algebraic one follows by thinking in terms of chains instead.

**Notation** Let $X$ be a pointed space, $*$ the base point, let $\pi$ be a subgroup of $\Sigma_n$, \[ X_n = K(R,n), \] $R$ a fixed field.

Since $X$ is pointed, $X^n$ is filtered:
\[
X^n = \{ (x_1, \ldots, x_n) | \text{at most 1 } x_i \neq * \} \subseteq \cdots \subseteq \{ x_1, \ldots, x_n \} | \text{at most } n-1 \} \subseteq X^n
\]

\[ F_0 = \bigwedge^n X \]

\[ F_{n-1} = \text{"fat wedge"} \]

The group $\pi$ acts on $X^n$:
\[
\sigma( (x_1, \ldots, x_n) ) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})
\]

\[ \dagger \] the group action respects the filtration (Say $F_i \to F_{i+1}$ is $\pi$-equivariant).

⇒ cofibers get a $\pi$-action.

Most important is

**Def** The $n$-fold smash power of $X$ is
\[
X^{(n)} = X^n / F_{n-1}
\]

This is canonically a $\pi$-space. However, the $\pi$-action isn't free \[ (\pi(x, \ldots, x) = (x, \ldots, x)) \].

Now we need an equivariant construction.

**Def** If $X$ is a $G$-space, then the Borel construction is the orbit space
\[
X_{hG} := EG \times_G X = (EG \times X)_G
\]

Here $EG$ is a free, contractible $G$-space \[ \dagger \] $G$ acts on $EG \times X$ diagonally: \[ g(e, x) = (g(e), g(x)). \]
Prop 4.1 If $G$ acts freely on $X$, then $X_{na} = X_G$.

If $G$ acts trivially on $X$, then $X_{hG} = BG 	imes X$.

Idea is that if $G$ acts freely, then $EG \times X \to X$ is $G$-equivariantly a homotopy equivalence.

Can realize $EG \times X$ as a bundle over $BG$:

1. $EG \times X \to X$ is a $G$-map $\Rightarrow$ get a map
   $$EG \times X \to EG \times X = BG.$$  
   This is the map

2. $G \to EG \xrightarrow{BG} \text{is a fibration. We form } EG \times X \text{ by replacing}$
   
   the fiber $G$ with $X$ using the $G$-action.

$$\Rightarrow X \to EG \times X \downarrow_{BG}$$

We can apply all of this to $X^{(n)} \downarrow \pi$.

$\Rightarrow$ Have bundles over $B\pi$ w/ total space
$$E_\pi \times F_{n-1} \downarrow \pi \times X^n.$$  

Since $F_{n-1} \leq X^n$ equivariantly,

$$E_\pi \times F_{n-1} \leq E_\pi \times X^n \text{ as a subbundle.}$$

Def. The $\pi$-extended power of $X$ is

$$D_\pi X = \left( E_\pi \times X^n \right) \left/ E_\pi \times F_{n-1} \right.$$  

We can rewrite the right-hand side as

$$E_\pi \left/ \pi \right. \downarrow \pi \times X^{(n)}.$$  

The $\pi$-extended power construction is the source of all Steenrod $\ast$ power operations.

We need to understand the cohomology, especially in the universal
case of $K_n$.

**Thm 1.** If $\overline{H}^r(x) = 0$ for $r < q$, then $\overline{H}^s(D, X) = 0$ for $s < q$.

Moreover, $\overline{H}^q(D, X) = (\overline{H}(x)^{\otimes n})^\pi$ invariant.

Can give a few proofs. For the first part, let's assume $X$ is s.c.

- The $q$-skeleton of $X$ is a wedge of $q$-spheres.
- The $nq$-skeleton of $X^{(n)}$ is a wedge of $(nq)$-spheres.
- The $nq$-skeleton of $D, X$ is a wedge of $(nq)$-spheres.

To better understand $\overline{H}^q(D, X)$, we use the relative Serre SS:

**Thm 2.** If $F' \subseteq F$ are a bundle over a fixed base $B$, then there is a $1^s$-quadrant SS with

$$E_2^{p, q} = H^p(B; \overline{H}^q(F, F'; R)) \Rightarrow H^{p+q}(E, E'; R).$$

In our case, $E' = E \hat{\times} F_{n-1}$, $E = E \hat{\times} X^n$, so $H^{p+q}(E, E'; R) = \overline{H}^{p+q}(D, X'; R)$.

Let $B = B_{\pi}$ has $\pi_{\pi} = \pi$. This acts non-trivially on $\overline{H}^*(F, F')$.

Now $H^*(F, F'; R) = \overline{H}^*(F/F'; R) = \overline{H}^*(X^{(n)})$

$= \overline{H}^*(x)^{\otimes n}$, this is all $\pi$-equivariant. So we know that $H^*(F, F'; R) = 0$, $H^*(x)^{\otimes n} = n^q$.

$$\begin{array}{c}
\overline{H}^*(\pi; \overline{H}^q(x)^{\otimes n}) \\
\hline
n^q \\
\hline
\end{array}$$
So for $s < n \cdot q$, $E_2^{s, s} = 0 \Rightarrow$

1. everything in $E_2^{s, s}$ is a perm cycle
2. $\overline{H}^\ast(D_\pi X) = \begin{cases} 0 & \ast < n \cdot q \\ H^\ast(\Pi_j \overline{H}(x)^{\otimes n}) & \ast = n \cdot q \end{cases}$

Since $H^0(G, M) = M^G$ (by def), we conclude

$$\overline{H}^\ast(D_\pi X) = \begin{cases} 0 & \ast < n \cdot q \\ \left(H^\ast(x)^{\otimes n}\right)_\Pi & \ast = n \cdot q \end{cases}$$

We apply this to $K_q$.

**Prop 2** $H^\ast(K_q; R) = \begin{cases} 0 & \ast < q \\ R & \ast = q \end{cases}$

**Cor** $\overline{H}^{n q}(D_\pi K_q; R) = R$, generated by a class $p_\pi L_q$ s.t. $p_\pi L_q \mapsto \overline{\Lambda}_q^{\otimes n} \overline{H}^{n q}(D_\pi K_q) \mapsto \overline{H}^{n q}(K_q^\ast)$

So in fact we have a map $D_\pi K_q \xrightarrow{p_\pi L_q} K_q$

**Def** Let $u \in H^q(x)$. Then the Total Steenrod power on $u$ is the composite

$$D_\pi X \xrightarrow{D_\pi u} D_\pi K_q \xrightarrow{p_\pi} K_q$$

To get Steenrod ops in the usual form, we pull back along the diagonal $X \xrightarrow{\Delta} X^\ast$ is $\pi_\ast$ equivariant $\Rightarrow$

$$E_{\pi_\ast} \overline{H}^\ast X \xrightarrow{\Delta} D_\pi X$$

$$B_{\pi_\ast} \overline{H}^\ast X$$

Now $\overline{H}^\ast(B_{\pi_\ast} \overline{H}^\ast X) = H^\ast(B_{\pi_\ast}) \otimes H^\ast(x)$, so

The composite $B_{\pi_\ast} \overline{H}^\ast X \xrightarrow{\Delta} D_\pi X \xrightarrow{p_{\pi_\ast} u} K_q$

is a sum $\sum b_i \otimes x_i$

$H^\ast(B_{\pi_\ast}) \otimes H^\ast(x)$
Let's restrict attention to $n = 8$, $\pi = \Sigma^2$, $\mathbb{R} = \mathbb{F}_2$.

Then $B\pi = \mathcal{R}P^\infty$, so $H^*(B\pi; \mathbb{F}_2) = \mathbb{F}_2[x]$, $|x| = 1$.

**Def** $\Delta^*(\mathbb{P}_2 L_q) = \sum x^iq_i \otimes Sq^i(u)$

This canonically defines elements $Sq^i(u) \in H^{2+i}(X; \mathbb{F}_2)$.

**Prop** $Sq^2 u = u^2$

**PF** Consider the "inclusion of the fiber" $K_q \hookrightarrow E_{\pi_+}^\pi K_q$

Then we have a square:

\[
\begin{array}{ccc}
E_{\pi_+}^\pi K_q & \xrightarrow{\Delta} & D_2 K_q \\
\uparrow \quad \quad \quad \uparrow \\
K_q & \xrightarrow{\Delta} & K_q^{(2)}
\end{array}
\]

and a map $D_2 K_q \rightarrow K_q^{(2)}$.

We look at what happens to $\mathbb{P}_2 L_q$ under these maps:

\[
\begin{array}{ccc}
\sum x^q \otimes Sq^i L_q & \xleftarrow{\text{Def of } Sq^i} & \mathbb{P}_2 L_q \\
\downarrow & & \downarrow \\
H^q(B\pi_+ K_q) & \xleftarrow{\text{Thm 1}} & H^q(D_2 K_q) \\
\downarrow & & \downarrow \\
Sq^q L_q & \xleftarrow{\text{def of } \Delta^2} & L_q^{(2)} \\
\downarrow & & \downarrow \\
L_q & \xrightarrow{\Delta} & L_q
\end{array}
\]

So $Sq^2 L_q = L_q^2$. Naturality gives the result.