Def/Axioms For each \( i \geq 0 \), there is a natural transformation
\[
\text{Sq}^i : H^n(x,A; \mathbb{F}_2) \to H^{n+i}(x,A; \mathbb{F}_2)
\]
s.t.
1. \( \text{Sq}^0 = \text{Id} \)
2. If \( i > 1 \), then \( \text{Sq}^i x = 0 \)
3. If \( i = 1 \), then \( \text{Sq}^i x = x^2 \)
4. If \( \delta \) is the conn. hom in the LES for the pair \((x,A)\), then
   \[
   \delta \text{Sq}^i = \text{Sq}^i \delta
   \]
5. (Cartan Formula) \( \text{Sq}^n(x,y) = \sum_{i+j=n} \text{Sq}^i x \cdot \text{Sq}^j y \)
6. (Adem Relation) If \( a < 2b \), then
   \[
   \text{Sq}^a \text{Sq}^b = \sum_{c=0}^{\lfloor \frac{b}{2} \rfloor} (b-c-1) \binom{a-2c}{c} \text{Sq}^{a+b-c} \cdot \text{Sq}^c
   \]

These are natural transformations, but we can tie this to our story.

Prop 1 \( H^n(x; G) = [x, K(G,n)] \)

This is a basic consequence of obstruction theory. Quick primer:
1. If \( g : X^{(n)} \to Y \), \( \forall n \) for every \((n+1)\)-cell \( \sigma \), we have an element of \( \pi_n(Y) \) given by \( g \circ \delta(\sigma) \). This gives an \((n+1)\)-cochain on \( X \) with coefficients in \( \pi_n Y \).

Obvious fact: There is an extension \( g = n \) of \( g \) over \( X^{(n+1)} \) iff this cochain vanishes.

The obstruction cochain is actually a cocycle \( i \) in some story is understanding the corresponding cohomology class.

There is a fundamental class \( i_n \in H^n(K(\pi,n); \mathbb{F}_2) \) s.t.
\[
f \in [X, K(\pi,n)] \leftrightarrow f^* i_n
\]
Since cohomology is representable, the Yoneda lemma gives an equiv
\[ \text{Nat}(\mathcal{H}^n(-; \mathbb{R}), \mathcal{H}^m(-; \mathbb{G})) = \mathcal{H}^m(\mathbb{K}(\pi, n); \mathbb{G}). \]

Can easily understand this Qly.

1. \(\mathbb{K}(\mathbb{Q}, 1) = S^1_{\mathbb{Q}} = \) mapping telescope \( S_1 \xrightarrow{2} S_1 \xrightarrow{w} S_1 \rightarrow \ldots \rightarrow S_1 \rightarrow \ldots \)
Universal coefs ensures that \( H^*(\mathbb{K}(\mathbb{Q}, 1); \mathbb{Q}) = E(x_1). \)

2. \(\mathbb{K}(\mathbb{Q}, 2) = CP_\mathbb{Q}^\infty \cap H^*(\mathbb{K}(\mathbb{Q}, 2); \mathbb{Q}) = \mathbb{Q}[x_2] \)

3. Have fibrations \(\mathbb{K}(\mathbb{Q}, n-1) = S_2 \mathbb{K}(\mathbb{Q}, n) \rightarrow \mathbb{K}(\mathbb{Q}, n) \)
so by induction, can compute \( H^*(\mathbb{K}(\mathbb{Q}, n)) \)

**Prop 2.** As an algebra \( H^*(\mathbb{K}(\mathbb{Q}, 2n+1)) = E(x_{2n+1}) \)
\( H^*(\mathbb{K}(\mathbb{Q}, 2n)) = \mathbb{Q}[x_{2n}] \)

When we use the Yoneda Lemma, this says the only natural transformations of \( H^*(-; \mathbb{Q}) \) are the cup-powers.
Want integral information, so we'll look at each prime. \( \Rightarrow \) Steenrod ops.
Our method, though, will be essentially the same as in the \( \mathbb{Q} \) case.
At \( p=2 \), the Steenrod algebra is fundamental here.

**Def.** The Steenrod algebra is the graded \( \mathbb{F}_2 \) algebra generated by classes \( Sq^i \) subject to the Adem relations. Denote this \( A \).
Pause here to describe another form of the Adem relations:

**Prop 3.** \( Sq^{2n-1} Sq^n = 0 \)

The others all follow from this by using Pascal's triangle:
\[
\begin{align*}
Sq^{2n-1} Sq^n &= 0 \\
Sq^{2n-2} Sq^n &= 0 \\
Sq^{2n-3} Sq^n &= 0 \\
Sq^{2n-2} Sq^n &= 0 \\
&+ Sq^{2n-1} Sq^{n-1} = 0 \\
&+ Sq^{2n-2} Sq^{n-2} = 0
\end{align*}
\]
This will follow easily from our analysis of the dual algebra and the cap product.

\[
\begin{align*}
\text{Ex:} & \quad Sq^3 Sq^2 = 0 \\
& \quad Sq^2 Sq^2 + Sq^3 Sq^1 = 0 \\
& \quad Sq^1 Sq^2 + Sq^3 = 0
\end{align*}
\]

Prop 4 \quad Sq^1 Sq^{2n} = Sq^{2n+1}

Prop 4 \quad Sq^1 Sq^{2n} = 0

\textbf{Def} The total squaring operation \( Sq : H^* \to H^* \) is defined by

\[ Sq(x) = \sum_{i=0}^{2n} Sq^i(x). \]

The Cartan formula ensures this is a ring hom.

Though this is not homogeneous, the homogeneous parts record the various steenrod actions.

\textbf{Ex:} \quad RP^\infty \quad H^*(RP^\infty; \mathbb{F}_2) = \mathbb{F}_2 [x_1]

\[
\begin{align*}
\text{By axioms (1-3),} & \quad Sq(x_1) = x_1 + x_1^2 \\
\text{So} & \quad Sq(x_1^2) = (Sq(x_1))^2 = x_1^2 + x_1^4 = x_1^2 + 0 + x_2^2 \\
& \quad Sq^0 \quad Sq^1 \quad Sq^2 \\
\text{Sq}(x_1^3) & \quad (Sq(x_1))^3 = (x_1 + x_1^2)(x_1^2 + x_1^4) = x_1^3 + x_1^5 + x_2^2 \\
& \quad Sq^0 \quad Sq^1 \quad Sq^2 \quad Sq^3
\end{align*}
\]

More generally,

Prop 5 \quad Sq^i(x^k) = \binom{i}{k} x^{k+i}

We can prove this by computing binom coeffs by writing out the dyadic (=binary) expansions of \( k \) and \( i \). If \( i = 0 \) in any place \( k = 0 \), the binom coeff is zero.

\textbf{Ex:} \quad CP^\infty \quad H^*(CP^\infty) = \mathbb{F}_2 [x_2] \Rightarrow Sq(x_2) = x_2 + 0 + x_2^2

So same analysis for \( TRP^\infty \) works here!
Some notation

Def: If $I = (i_1, i_2, \ldots)$ is a finite sequence of pos. ints, let $\text{Sq}^I = \text{Sq}^{i_1} \cdot \text{Sq}^{i_2} \cdots$

Say $I$ is admissible if $i_j > 2i_{j-1}$

If $I$ is admissible, let $|I| = \sum_{j=1}^{\infty} i_j$ be the degree

$e(I) = 2i_1 - |I| = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots$, denote the excess.

Prop: $\text{Sq}^I$ raises degree by $|I|

1. If $|x| < e(I)$, then $\text{Sq}^I(x) = 0$

2. If $|x| = e(I)$, then $\text{Sq}^I(x) = (\text{Sq}^I(x))^2$ for a subsequence.

is obvious, 2 will follow from our analysis of the SS for $K(\mathbb{H}/n)$.

Thm: $\text{Sq}^I$ for $I$ admissible forms a basis for $A$.

This will also follow. We check this on test spaces like $K(\mathbb{H}/n)$ or $(\mathbb{RP}^n)^n$.

We'll finish with the dual. Since $A$ is locally finite (each degree is finite dim), $A^*$ is well behaved.

Prop 2: The Cartan formula $\text{Sq}^i \rightarrow \sum_{j+k=i} \text{Sq}^j \otimes \text{Sq}^k$ induces a coproduct on $A$ that is an alg hom.

This makes $A$ into a cocommutative Hopf algebra. $A^*$ is a commutative Hopf alg.

Thm 2 (Milnor): $A^* = \bigoplus \mathbb{Z}[x_j, y_j, \ldots]$, $|x_j| = 2^j - 1$

$\Delta(x_j) = \sum_{j+k=\ell} x_j^k \otimes y_k$

The class $x_k$ is dual to $\text{Sq}^I_k$, $I_k = (2^k, 2^k, \ldots, 2, 1)$

Capping with $x_1$ gives the easier Adem relations.