Look today at the Serre SS for the case $F=S^n$
So if $S^n \rightarrow E \rightarrow B$ is a (simple) spherical fibration, then the
Serre SS is especially simple: it is a long exact sequence

\[ H^*(B) \otimes H^n(S^n) \rightarrow \text{All Zero!} \rightarrow H^*(E) \]

As an algebra, $E_2 = H^*(B) \otimes H^*(S^n) = H^*(B) \otimes E(x_n)$
Since $H^*(B)$ sits on the zero line, everything in $H^*(B)$
is a permanent cycle. $\Rightarrow$ differentials are completely determined
by those on $x_n$.

For degree reasons, only possibility is a $d_{n+1}$.
So $d_{n+1}(x) = e \in H^{n+1}(B)$, and if $b \in H^*(B)$,
\[ d_{n+1}(x \cdot b) = d_{n+1}(x) \cdot b + (-1)^{x_1} x \cdot d_{n+1}(b) \]
\[ = e \cdot b. \]
Thus $H^{*q}(E_{n+1}, d_{n+1}) = \begin{cases} \text{ker}(H^*(B) \cdot e \rightarrow H^*(B)) & \text{in } q=n, \\ 0 & \text{in } q=0, n, \\ \text{coker}(H^*(B) \cdot e \rightarrow H^*(B)) & \text{in } q=0 \end{cases}$

For degree reasons, this is also $E_\infty$.
We'll return to reassembling this into $H^*(E)$ in a minute.

The class $e$ is called the Euler class of the spherical fibration.

This notation comes from vector bundles & characteristic classes.
If $V \rightarrow B$ is a vector bundle w/ a metric (say $B$ compact enough)
then we have an associated sphere bundle $S(V) \rightarrow B$
whose fiber over $b$ is the unit sphere of $V_b$. 
Then the class $\xi$ for the same $SS$ is the Euler class for the vector bundle $V$. (If $V$ is the tangent bundle to a manifold $M$, the $\xi = (\text{Euler characteristic}) \cdot \text{Poincaré duality class}$)

**Ex:** $H^*(V_2(\mathbb{R}^{n+1}))$.

**Def** $V_2(\mathbb{R}^{n+1}) =$ space of orthogonal pairs of unit vectors in $\mathbb{R}^{n+1}$

So we can identify $V_2(\mathbb{R}^{n+1})$ with the unit sphere bundle to $T(S^n)$: $T(S^n) = \left\{ (x, \bar{v}) \in (\mathbb{R}^{n+1})^2 \mid \frac{x \cdot x}{x} = 1, \frac{x \cdot \bar{v}}{\bar{v}} = 0 \right\}$

so the unit sphere bundle is $S(T) = \left\{ (x, \bar{v}) \mid \frac{x \cdot x}{x} = 1, \frac{x \cdot \bar{v}}{\bar{v}} = 0 \right\}$ has fiber $S^{n-1}$ over $S^n$.

**Euler Class:** $(1 + (-1)^n)[S^n]$.

$n=2$:

Euler class: $d(x_1) = z y_2$

$E_3$:

\[ H^*(V_2(\mathbb{R}^3)) = \left\{ \begin{array}{ccc} \mathbb{Z}_4 & 3 \\ \mathbb{Z}/2 & 2 \\ 0 & 1 \end{array} \right. \]

We should expect this: Given $\underline{x}, \underline{v}$ s.t. $\underline{x} \cdot \underline{v} = 0$, $||\underline{x}|| = ||\underline{v}|| = 1$, there is a unique $\underline{w}$ s.t. $[\underline{x}, \underline{v}, \underline{w}] \in SO(3)$.

So $V_2(\mathbb{R}^3) = SO(3) = \mathbb{R}P^3$

this is done by looking at the conjugation action of the unit quaternions on the imaginary ones.
n=3: \[ S^2 \to V_2(\mathbb{R}^n) \]
\[ \frac{S^3}{S^3} \]

\[ e = 0, \text{ so} \]
\[ E_2 = E_\infty \]

\[ H^*(V_2(\mathbb{R}^n)) = \left\{ \begin{array}{ll}
\mathbb{Z} & n = 5 \\
\mathbb{Z} & n = 4 \\
\mathbb{Z} & n = 2 \\
0 & n = 0 
\end{array} \right. \]

In general, \[ H^*(V_2(\mathbb{R}^{n+1})) = \left\{ \begin{array}{ll}
E_{n-1}(x_{n-1}, x_n) & n \text{ odd} \\
\left\{ \begin{array}{c}
\mathbb{Z} \\
0 \\
\mathbb{Z}
\end{array} \right. & n \text{ even}
\end{array} \right. \]

Ex: Let \( S^1 \to E \to \mathbb{C}P^n \) be the spherical fibration with the class \( 2x \in H^2(\mathbb{C}P^n) \) (this is the sphere bundle associated to \( L^2 \to \mathbb{C}P^n \), \( L \) the canonical line bundle and \( E \approx \mathbb{R}P^n \))

So \[ E_3 : \]
\[ E_\infty \]

\[ \ast = \mathbb{Z}/2 \]

We can now return to the case of rebuilding the cohomology. First look at effects of the maps in the fibration.

Prop 1: \[ \pi^*(H^*(B)) \] is the subring of \( H^*(E) \) given by \( E_\infty^* \)

2. The image of \( \iota^*(H^*(E)) \) in \( H^*(F) \) is given by \( E_\infty^{*,*} \)

We can see this by comparing with the SSS for the fibrations \( F \to E \to * \) and \( * \to B \to B \).
A more careful analysis of the filtration gives the following.

**Prop 2** \( H^i(E) \) has a filtration \( F_i \) s.t.

\[
F_0 = E_\infty^0 \quad \downarrow \quad F_i/F_{i-1} = E_\infty^{-i,i}.
\]

So 
\[
E_\infty^0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \subseteq E_\infty^n.
\]

Now we piece together the Serre SS for \( S^\infty \to E \to B \) w/ euler class \( e \).

Already saw that we have a SES

\[
0 \to E_\infty^{k,n} \to H^k(B) \otimes H^n(S^\infty) \to H^k+n+1(B) \to E_\infty^{k,n+1} \to 0
\]

The previous props, together with sparceness, show that there is a SES

\[
0 \to E_\infty^{k,n} \to H^k(E) \to E_\infty^{k-n,n} \to 0
\]

Splicing these all together gives a long exact seq.

**Thm 1** If \( S^n \to E \to B \) is a simple fibration, then

\[
\ldots \to H^{k+n}(E) \to H^k(B) \xrightarrow{\pi^*} H^{k+n+m}(B) \to H^{k+n+m}(E) \to \ldots
\]

is exact, where \( e \) is the euler class.

There is a big case where we don't want simple fibrations:

**Def** Let \( B_\text{Top} : \text{Top} \to \text{Top} \) be the classifying space functor.

If \( G \) is an abelian group, can choose \( BG \) to be one as well. Moreover, \( \Pi_{k+1} BG = \Pi_k G \neq \emptyset \).

**Prop 3** If \( 1 \to N \to G \to H \to 1 \) is a SES, then \( BN \to BG \to BH \) is a fibration.
Remark. In fact, \( G \rightarrow H \rightarrow BN \), etc are also fibrations. This is very often not simple.

**Def.** \( H^*(G) = H^*(BG) \).

We can incorporate non-trivial \( G \)-modules by looking at bundles over \( BG \) \( \triangleright \) homology w/ twisted coefficients.

The Hochschild-Serre SS is the SS associated to the fibration \( BN \rightarrow BG \rightarrow BH \):

\[
H^p(H; H^q(N; M)) \Rightarrow H^{p+q}(G; M)
\]

\( M \) a \( G \)-module, \( H^q(N; M) \) an \( H \)-module by twisting.

**Ex.** \( H^*(\mathbb{Z}_3; \mathbb{Z}_{(3)}) \).

Need some facts.

1. \( \mathbb{Z}/3 \rightarrow \mathbb{Z}/3 \rightarrow \mathbb{Z}/2 \) is exact
2. \( \mathbb{Z}/2 \) acts on \( \mathbb{Z}/3 \) by inversion.
3. If \( p \) is a unit in \( R \), then \( H^{*\geq 0}(\mathbb{Z}/p; R) \Rightarrow 0 \)

In our case, \( 3 \) is satisfied. So the HS SS takes the form

\[
E_2^{p,q} \Rightarrow H^{p+q}(\mathbb{Z}/3; \mathbb{Z}_{(3)})
\]

1. If \( p > 0 \), then \( E_2^{p,q} = 0 \).

For \( p = 0 \), \( H^*(\mathbb{Z}/3; \mathbb{Z}_{(3)}) = \mathbb{Z}_{(3)}[x]/3x \) \( \triangleright \mathbb{Z}/2 \) acts by \( x_2 \mapsto -x_2 \). So \( H^0(\mathbb{Z}/2; \mathbb{Z}_{(3)}[x]/3x) = \mathbb{Z}_{(3)}[x^2]/3x^2 \) and this is also \( E_\infty ! \)