Filtrations & Spectral Sequences

Def A filtration is a functor from the poset category of the integers to another category.

The other category is normally built out of R-mod.

Remark There is an issue of variance. A covariant functor gives an "increasing" filtration. A contravariant functor gives a "decreasing" filtration.

Key Example:

Filtered R-modules: $R\text{-}Mod_{inj} = \begin{cases} R\text{-modules} \\ \text{injective homs} \end{cases}$

Functor $\mathbb{Z} \xrightarrow{F} R\text{-}Mod_{inj}$ $\longleftrightarrow$ a sequence of R-modules $F_n M$ with compatible injections $F_n M \hookrightarrow F_m M$.

$\longleftrightarrow$ a sequence of submodules

$$\cdots \subseteq F_n M \subseteq F_{n+1} M \subseteq \cdots \quad M = \bigcup_{n \in \mathbb{Z}} F_n M = \varinjlim F_n M$$

Remark This is the usual definition, and we will use the interchangably.

Ex: $R = M = \mathbb{Z}$

$$F_n M = \begin{cases} \mathbb{Z} & n \geq 0 \\ \cup\!\mid \\ 2^{-n}\mathbb{Z} & n \leq -1 \end{cases}$$

This works more generally: If M is an abelian gp, then there is a filtration $\cdots \subseteq m^2 \cdot M \subseteq m \cdot M \subseteq M \subseteq M \cdots$ for any $m \in \mathbb{Z}$.

Ex: $R = k[x] = M$

$p(x) \in R$ $\qquad F_n M = \begin{cases} k[x] & n \geq 0 \\ p(x)^{-n} \cdot k[x] & n \leq -1 \end{cases}$.

Def A map of filtered somethings is a natural transformation of the functors.

In the filtered module context, this is the same thing as a homomorphism of R-modules $M \longrightarrow N$ s.t.

$$f(F_n M) \subseteq F_n N.$$

Associated to a filtration (in an abelian category) is a graded object.

Def The associated graded $Gr(M)$ is defined by

$$Gr_n(M) = F_n M / F_{n-1} M.$$

Passage to associated graded is obviously functorial.

Ex: $R = \mathbb{Z} = M$, $F_n M = \begin{cases} 2^{-n}\mathbb{Z} \\ \mathbb{Z} \end{cases}$, then

$$Gr_n M = \begin{cases} 0 & n > 0 \\ \mathbb{Z}/2 & n \leq 0 \end{cases},$$ and this holds generically.

For us, the most important thing is a filtered DGM.

$\longleftrightarrow$ A sequence $(F_n M, d_n)$ of DGMs w/ compatible inclusions $F_n M \longrightarrow F_m M$ s.t.

$$\begin{tikzcd} F_n M \arrow[r] \arrow[d, "d_n"] & F_m M \arrow[d, "d_m"] \\ F_n M \arrow[r] & F_m M \end{tikzcd}$$

commutes.

$\longleftrightarrow$ A DGM (M, d) together with a filtration of M s.t. $d(F_n M) \subseteq F_n M$.

$\longleftrightarrow$ A filtered graded module w/ a map d of filtered graded modules with $d^2 = 0$ and d shifts degree by ± 1

So the associated graded of a DGM is a DGM!

If we filter things right, it will be easier to understand.

Guiding Principle:

Filter a DGM so that we can compute with the AG.

Guiding Question:

How can we recover M from the AG?

Ex: $M_\bullet = \quad \underset{1}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{0}{\mathbb{Z}}$ So $Gr_p(M_q) =$

$F_{n\geq 0} M = \underset{1}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{0}{\mathbb{Z}}$

$F_{-n} M = 2^n \mathbb{Z} \xrightarrow{\cdot 2} 2^n \mathbb{Z}$

$$\begin{array}{c|cc} p \backslash q & 1 & 0 \\ \hline >0 & 0 & 0 \\ 0 & \mathbb{Z}/2 \xrightarrow{0} & \mathbb{Z}/2 \\ -1 & \mathbb{Z}/2 \xrightarrow{0} & \mathbb{Z}/2 \end{array}$$

Why is $Gr(d) = 0$? Check the base case $\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \\ \cup & & \cup \\ 2\mathbb{Z} & \xrightarrow{\cdot 2} & 2\mathbb{Z} \end{array}$

The differential on Gr is induced by the composite $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} = F_0/F_{-1}$ and this is zero.

So $Gr(M_\bullet)$ is bigraded, but the differential is zero!

<u>Remark</u> The sketch above shows how to compute differentials/functions on the AG. Given $f: M \longrightarrow N$, $Gr_n f$ is defined by taking the composite

$$\begin{tikzcd} F_n M \arrow[r] & F_n N \arrow[r] \arrow[d, phantom, "\circlearrowright"] & F_n N / F_{n-1} N \\ F_{n-1} M \arrow[u, hook] \arrow[rr, mapsto] & & \{0\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Since $f(F_{n-1}M) \subseteq F_{n-1}N$,

So we get a canonical map $F_n M / F_{n-1} M \xrightarrow{Gr_n f} F_n N / F_{n-1} N$

The problem?

Associated graded doesn't commute with homology:

$$Gr_*(H_\bullet) \not\cong H_\bullet(Gr_*).$$

This is obvious from the previous example.

The solution? Spectral Sequences. Normally, we will have a SS $\quad H_\bullet(Gr_*) \Rightarrow Gr_*(H_\bullet)$.

<u>Def</u> A cohomological, first quadrant spectral sequence is a sequence of bigraded DGMs $E_n^{p,q}$ s.t.

1. $d_n : E_n^{p,q} \longrightarrow E_n^{p+n,q+1-n}$
2. $E_{n+1} = H^{\bullet,\bullet}(E_n)$.
3. $E_n^{p,q} = 0$ if $p<0$ or $q<0$

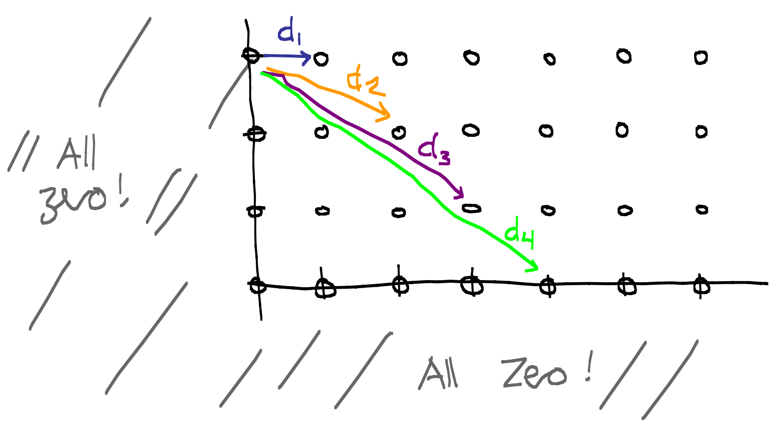
The first part just says that d_n is "degree 1" in the total degree $r=p+q$. This seems very contrived, but we'll see how this plays out quite naturally.

There is an obvious homological version. There $d_n : E^n_{p,q} \to E^n_{p-n,q+n-1}$.

We say that an element x s.t. $d_n(x)=0$ for all n is a permanent cycle. Elements in $\mathrm{Im}(d_n)$ are boundaries

The first quadrant condition shows that there are only finitely many possible non-zero differentials on any given class.

Generic pictures:



So from $(0,3)$, can have only d_1, d_2, d_3, d_4

We'll actually have extra structure normally.

<u>Def</u> $E_\bullet^{\bullet,\bullet}$ is a spectral sequence of algebras if $E_n^{*,*}$ is a DGA and $E_{n+1}^{*,*} = H^{*,*}(E_n)$ as algebras.

Here we use the "total degree" in the Leibnitz Rule.