Derived Functors: Ext & Tor

Note Title 1/26/2009

Derived functors measure the failure of a functor to preserve exactness. Homology does something similar, so it's no surprise they are related.

Rather than focus on the general theory, we will focus on the two most important examples: Ext (from Hom) and Tor (from $\otimes$)

I. Hom & Ext

Fix a ring R and an R-mod M.

<u>Lemma 1</u> If $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is an exact seq of R-modules, then
$$0 \to \mathrm{Hom}_R(M,A) \xrightarrow{\varphi_*} \mathrm{Hom}_R(M,B) \xrightarrow{\psi_*} \mathrm{Hom}_R(M,C)$$
is an exact sequence of abelian groups.

<u>Pf</u>: We have to show that φ_* is injective & $\mathrm{Im}(\varphi_*) = \ker(\psi_*)$.

$\ker(\varphi_*) = \{0\}$: If $\varphi_* f = 0$, then $\varphi(f(m)) = 0$ for all $m \in M$. Since φ is injective, this means $f(m) = 0$ for all m, so $f = 0$.

$\ker(\psi_*) = \mathrm{Im}(\varphi_*)$: The inclusion $\supseteq$ is easy: $\psi_*(\varphi_* f) = (\psi \circ \varphi)_* f = 0_* f = 0$ by exactness.

The other inclusion is also easy: If $\psi_* g = 0$, then by exactness, $g(m) \in \mathrm{Im}(\varphi)$ for all m. Since φ is an isom onto its image, $g = \varphi_*(\varphi^{-1} g)$. ▨

It is <u>not</u> the case that $\mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C) \longrightarrow 0$ is always exact.

<u>Ex</u>: $R = \mathbb{Z}$, $A = B = R$, $M = C = \mathbb{Z}/2$. Then
$$\underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z})}_{= 0} \longrightarrow \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)}_{= \mathbb{Z}/2}$$
is not surjective.

Def We say $\mathrm{Hom}_R(M,-)$ is a <u>left exact functor</u>, since it takes exact sequences to left exact ones.

Lemma 2: If $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is an exact seq. of R-modules, then $$0 \to \mathrm{Hom}_R(C,M) \xrightarrow{\psi^*} \mathrm{Hom}_R(B,M) \xrightarrow{\varphi^*} \mathrm{Hom}_R(A,M)$$ is exact.

Pf The $(-)^*$ is precomposition, and the arguments are dual to those above.

$\ker(\psi^*) = \{0\}$: If $\psi^* f(b) = 0$ for all $b \in B$, then $f(\psi(b)) = 0$ by definition. Since every $c \in C$ is $\psi(b)$ for some b, $f(c) = 0$ for all $c \in C \Rightarrow f = 0$.

$\ker(\varphi^*) = \mathrm{Im}(\psi^*)$: $\supseteq$ is again functoriality: $\varphi^* \circ \psi^* = (\psi \circ \varphi)^* = 0^* = 0$.

$\subseteq$) Let $g: B \to M$ be such that $\varphi^* g = 0$. Thus g descends to the quotient $B/\mathrm{Im}(\varphi) = B/\ker(\psi)$. The latter is can. isom to C via ψ, so we get a map $f: C \xrightarrow{\psi^{-1}} B/\ker\psi \xrightarrow{\overline{g}} M$ s.t. $\psi^*(f) = g$. ☒

Oddly enough, we also say that $\mathrm{Hom}_R(-,M)$ is left exact. Again: no right exactness. Let $R = \mathbb{Z}$, $M = \mathbb{Z}$. Then

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$ is exact, but

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) & \xrightarrow{(\cdot 2)^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow{\quad 2 \quad} & \mathbb{Z} \end{array}$$ is not surjective.

Remark In lemmata 1 & 2, the left exact sequences are sequences of $Z(R)$-modules, where $Z(-)$ denotes the center of the ring. If R is commutative, then $R = Z(R)$, so everything is still an R-mod.

We want to measure the failure of Hom_R to be right exact in a functorial way. We first note that there are modules for which $\mathrm{Hom}_R(M,-)$ is exact:

Lemma 3 If M is a free R-module $(\cong \bigoplus_{i \in I} R$ for some $I)$, then $\mathrm{Hom}_R(M,-)$ is an exact functor.

This is because we can specify an R-linear map out of M

by specifying it on a basis of M. This is identical to what happens for vector spaces. We need to slightly fatten up this class (it makes some arguments "smaller").

Def An R-module P is <u>projective</u> if $\mathrm{Hom}_R(P,-)$ is exact.

This means that given a surjective map $M \twoheadrightarrow N$ and a map $P \to N$, we can find a map $P \to M$ making

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ M \arrow[r, two heads] & N \end{tikzcd}$$

commute.

Prop 1 Projective modules are summands of free modules.

We can compute the derived functors of $\mathrm{Hom}_R(M,-)$ by replacing M with projective ones.

Def A <u>projective resolution</u> of M is an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

where each P_i is projective.

If we drop the "augmenting" M, then we get a dgm with each homogeneous piece projective & w/ $H_*(P_\bullet) = \begin{cases} 0 & * \neq 0 \\ M & * = 0 \end{cases}$.

We can use a projective resolution to compute the derived functors:

Def $$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}(P_\bullet, N))$$

Here N is a dgm concentrated in degree 0 w/ $d_N = 0$.

<u>Ex</u> $M = \mathbb{Z}/p$. A good projective resolution is given by

$$\underset{P_1}{\mathbb{Z}} \xrightarrow{\cdot p} \underset{P_0}{\mathbb{Z}}$$

So if $N = \mathbb{Z}$:

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) = H^i(\mathrm{Hom}(P_\bullet, \mathbb{Z})) = H^i\left(\begin{array}{cc} {}_1 & \mathbb{Z} \\ & \uparrow \cdot p \\ {}_0 & \mathbb{Z} \end{array}\right) = \begin{cases} \mathbb{Z}/p & i = 1 \\ 0 & i \neq 1 \end{cases}$$

if $N = \mathbb{Z}/p$:

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) = H^i\left(\begin{array}{cc} {}_1 & \mathbb{Z}/p \\ & \uparrow \cdot p \\ {}_0 & \mathbb{Z}/p \end{array}\right) = \begin{cases} \mathbb{Z}/p & i = 0,1 \\ 0 & i \neq 0,1 \end{cases}$$

Remark: If R is a PID, then $\mathrm{Ext}^i(M,N)=0$ for $i>1$ (submodules of frees are free)

If R is a division ring, then $\mathrm{Ext}^i(M,N)=0$ for $i>0$ (all R-modules are free).

Thm 1 ① $\mathrm{Ext}^i_R(M,N)$ is independent of the choice of resolution.

② $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$

③ If $0 \to A \to B \to C \to 0$ is exact, then

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to \mathrm{Ext}^1_R(M,A) \to \cdots$$

is long exact.

Pf Homework. ③ is the homology LES + ②. ☐

III. $\otimes$ & Tor. All essentially the same for the adjoint: Hom

Lemma 4: If $0 \to A \to B \to C \to 0$ is an exact sequence of Right R-modules, then

$$A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is exact.

The tensor product is <u>right exact</u>.

There is an immediate analogue for left R-modules & a right R-mod M. For us, R will usually be commutative.

Def An R-mod F is <u>flat</u> if $\otimes F$ is an exact functor.

Prop 2 Projective R-modules are flat.

Free R-modules are obviously flat & projectives are summands of frees.

Def $\mathrm{Tor}_i^R(M,N) = H_i(M \otimes_R P_\cdot)$ where $P_\cdot$ is a proj resolution of N.

Thm 2 ① Tor is independent of choice of $P_\cdot$.

② If R is commutative, $\mathrm{Tor}_i(M,N) \cong \mathrm{Tor}_i(N,M)$

③ $\mathrm{Tor}_0(M,N) = M \otimes N$ & If $0 \to A \to B \to C \to 0$ is exact,

$$\cdots \to \mathrm{Tor}_1(B,N) \to \mathrm{Tor}_1(C,N) \to A \otimes N \to B \otimes N \to C \otimes N \to 0$$ is exact.