Ext AND Tor HOMEWORK

Let k be a field and let $P_n(x) = k[x]/x^n$ be the truncated polynomial ring of height n over k. If $n = 2$, we'll also denote P_2 by $E(x)$, the exterior algebra generated by x.

(1) Find a projective (free) resolution of k as a $k[x]$-module. Use this to compute $Ext^*_{k[x]}(k,k)$, $Ext^*_{k[x]}(k,P_n(x))$, and the corresponding Tor groups.

(2) Prove that $Ext^i_{k[x]}(M,N) = 0$ for $i > 1$ and M, N finitely generated (though this last part doesn't really matter). (Hint: Consider what happens for $Ext_{\mathbb{Z}}$.) We say that $k[x]$ has cohomological dimension 1. Similarly, $k[x_1,\ldots,x_n]$ has cohomological dimension n.

(3) As a kind of converse to the previous problem (figure out what this statement means), show that if $R = k[x,y]$ and $M = (x,y)$ is the ideal generated by x and y, then M is not flat and hence not projective. Do this by finding a projective resolution of M and computing $Tor_1(M,k)$.

(4) Compute $Ext^*_{E(x)}(k,k)$, $Tor^{E(x)}_*(k,k)$, and the analogous groups for P_n.

(5) The group $Hom_R(M,M)$ is a ring for any R-module M. This fact "derives" to show that $Ext_R(M,M)$ is a ring for any M. Determine how the multiplication is defined compute the ring structure for the previous problem.

(6) The group $Ext^i_R(M,N)$ is related to isomorphism classes of extensions

$$0 \to N \to E_1 \to \cdots \to E_i \to M \to 0$$

(this is why it is called "Ext"). In this way, the ring structure on Ext is obvious: it's concatination. Read about or figure out the equivalence between our definition of Ext and the one via extensions.

(7) The group Tor measures "torsion" in a module. Show how this works for $R = \mathbb{Z}$ and more generally, if $a \in R$ is not a zero divisor, then see what happens with Tor for the short exact sequence $0 \to R \xrightarrow{a} R \to R/a \to 0$.

(8) If our ring is a graded ring, and all of the modules are graded modules, then the functors Ext^* and Tor_* become bigraded (this is immediate from considering graded projective resolutions). Compute the bigraded groups (and guess or prove the ring structure):

$$Ext^*_{E(x)}(k,k), Ext^*_{P_n(x)}(k,k), Ext^*_{k[x]}(k,k).$$

Your answer should depend on the degree of x, and the actual ring structure will also depend heavily on this and the characteristic of k. You may assume that Ext is graded commutative.