PROBLEM SET #4

DUE: NOVEMBER 15TH

This set will be run a little differently. Many of the problems will have a regular version and a "hard form". You can choose which ones you want to do (though the harder one subsumes the easier). Additionally: hard ones are worth 15 total points (so you can earn up to 5 extra credit points).

The Interplay between $\mathcal{L}(U,V)$ and $V\otimes U^*$.

Problem 1. *Show for $U=V=\mathbb{R}^2$ that $V\otimes U^*\to\mathcal{L}(U,V)$ described in class is an isomorphism. Show also that when we identify U^* with row vectors and V with column vectors that the isomorphism is "matrix multiplication":*

$$\vec{v}\otimes\vec{u}^t\mapsto\vec{v}\cdot\vec{u}^t.$$

Hard Form 1. *Show that for general finite dimensional U and V,*

$$V\otimes U^*\to\mathcal{L}(U,V)$$

is given by $\bar{v}\otimes f\mapsto f(-)\cdot\vec{v}$ and hence is also "matrix multiplication".

Problem 2. *Given a basis $\{\vec{u}_1,\dots,\vec{u}_n\}$ of U and $\{\vec{v}_1,\dots,\vec{v}_m\}$ of V, we get a basis $\{\vec{v}_i\delta_{\vec{u}_j}\}$ of $\mathcal{L}(U,V)$ by specifying*

$$\vec{v}_i\delta_{\vec{u}_j}(\vec{u}_k)=\begin{cases}\vec{v}_i & j=k\\ 0 & \textit{otherwise.}\end{cases}$$

Show that under the map

$$\mathcal{L}(U,V)\otimes\mathcal{L}(U',V')\to\mathcal{L}(U\otimes U',V\otimes V')$$

described in class, we have

$$(\vec{v}_i\delta_{\vec{u}_j})\otimes(\vec{v}_k'\delta_{\vec{u}_\ell'})\mapsto(\vec{v}_i\otimes\vec{v}_k')\delta_{\vec{u}_j\otimes\vec{u}_\ell'}.$$

Playing with $\mathcal{L}\big(U,\mathcal{L}(V,W)\big)\cong\mathcal{L}(U\otimes V,W)$. For the next few parts, Let $f\in\mathcal{L}\big(U,\mathcal{L}(V,W)\big)$ (so this is a linear transformation on U with values in the linear transformations from V to W.

Define $\hat{f}\colon U\times V\to W$ by

$$\hat{f}(\vec{u},\vec{v})=f(\vec{u})(\vec{v}).$$

Problem 3. *Show that $\hat{f}$ is bilinear.*

Now let $L\colon U\otimes V\to W$ be linear. Let $f_L\colon U\to\mathcal{L}(V,W)$ be defined by

$$f_L(\vec{u})(\vec{v})=L(\vec{u}\otimes\vec{v}).$$

Problem 4. *Show that if L_f is the map $U\otimes V\to W$ associated to the bilinear map $\hat{f}$, then the linear transformation f_{L_f} is f.*

Problem 5. *Show that L_{f_L} is L. Conclude that the two maps are inverses and thus the two spaces are [naturally] isomorphic.*

Hard Form 2. *Show that $\mathcal{L}(U, \mathcal{L}(V, -))$ and $\mathcal{L}(U \otimes V, -)$ are naturally isomorphic by showing that they represent the same collection of objects (naturally in the $-$ variable). In other words, show that (a) they give the same collection of objects for each value of W, and (b) given a map $W \to W'$, the identifications are compatible with the map.*

Determinants. This problem completes the proof of the "coordinate free" form of the determinant discussed in class.

Problem 6. *Verify for $n = 2$ or 3, and $V = \mathbb{R}^n$, and for $L \colon V \to V$ that $\Lambda^n(L)$ is multiplication by $|[L]_{\mathcal{B}}|$ for some (and therefore any) basis $\mathcal{B}$ of V.*

Hard Form 3. *Show for a general n-dimensional vector space V over $\mathbb{F}$ and $L \colon V \to V$ that $\Lambda^n(L)$ is multiplication by $|[L]_{\mathcal{B}}|$ for some (and again, for any) basis $\mathcal{B}$.*

Invariant Form of the Trace. The vector space $\mathbb{F}$ has a distinguished basis vector 1. We've seen that this gives an isomorphism

$$\mathcal{L}(\mathbb{F}, W) \cong W.$$

We can apply this to $W = \mathcal{L}(V, V) = V \otimes V^*$. There is a distinguished element in this vector space: the identity map $V \to V$. Thus we get a map

$$\mathbb{F} \to V \otimes V^*$$

that sends 1 to the identity. We call this map "coevaluation".

Problem 7. *Let $V = \mathbb{R}^3$. Describe the image of 1 under this map as an element of $V \otimes V^*$ and also as an element of $\mathcal{L}(V, V) = M_3(\mathbb{R})$.*

Hard Form 4. *Repeat this for $V = \mathbb{F}^n$.*

Problem 8. *Now consider the map $V \times V^* \to \mathbb{F}$ given by $(\vec{v}, f) \mapsto f(\vec{v})$. Show this is bilinear.*

Let $ev \colon V \otimes V^* \to \mathbb{F}$ be the corresponding linear map, and compute for $V = \mathbb{R}^3$ what this map does to a general element (viewed as a 3×3-matrix).

Hard Form 5. *Repeat this for an arbitrary finite dimensional V and $\mathbb{F}$. Conclude that the composite*

$$\mathbb{F} \to V \otimes V^* \to \mathbb{F}$$

is the dimension and is independent of choice of basis.

Since $V \otimes V^*$ is the same thing as $\mathcal{L}(V, V)$, given any $T \in \mathcal{L}(V, V)$ we get a map $T_* \colon V \otimes V^* \to V \otimes V^*$ given by composing with T.

Problem 9. *In the $V = \mathbb{R}^2$ case, after identifying $V \otimes V^*$ with $M_2(\mathbb{R})$, show that this amounts to matrix multiplication.*

Now we can define the trace: $Tr(T)$ is the composite

$$\mathbb{F} \xrightarrow{coev} V \otimes V^* \xrightarrow{T_*} V \otimes V^* \xrightarrow{ev} \mathbb{F}.$$

Since this is a linear map $\mathbb{F} \to \mathbb{F}$ it amounts to multiplication by a number. We will also denote this number $Tr(L)$.

Hard Form 6. *After choosing a basis, show this coincides with the usual notion of trace.*