PROBLEM SET #2

DUE OCTOBER 6

1. Basic Properties of the Determinant. In this problem, let A be an $n \times n$ matrix over $\mathbb{F}$.

(1) Using the definition, show that if A has two rows which are scalar multiples of each other (so for some i and k, $a_{i,j} = \lambda a_{k,j}$ for all j), then $|A| = 0$.
(2) Show that $|A| = |A^t|$.
(3) Again using the definition, show that if we swap two rows or columns in a matrix, then we swap the sign of the determinant (you may use without proof the following facts: $sign(\sigma \circ \tau) = sign(\sigma)sign(\tau)$, and if $f_{i,j}$ is the bijection from $\{1, \ldots, n\}$ to itself that swaps i and j and fixes everything else, then $sign(f_{i,j}) = -1$.)
(4) Show $A \cdot adj(A) = \det(A)I_n = adj(A) \cdot A$.
5* This problem is optional! Justify the results about the sign representation used in 3. The basic idea is that anything can be written as a product of transpositions $f_{i,j}$ (and as a fun pop-culture reference, this group and these ideas occurred in a recent episode of Futurama).

2. Let U, V, W be vector spaces over $\mathbb{F}$. This problem and the following will explore the "categorical" properties of finite direct sums=finite direct products.

(1) Let $\pi_U : U \oplus V \to U$ and $\pi_V : U \oplus V \to V$ be the projection maps $\pi_U(\vec{u} + \vec{v}) = \vec{u}$ (resp for V). Show that these are surjective linear transformations with $\ker(\pi_V) = U$ and $\ker(\pi_U) = V$.
(2) Show that we have a natural isomorphism
$$\mathcal{L}(W, U \oplus V) \to \mathcal{L}(W, U) \oplus \mathcal{L}(W, V)$$
given by $T \mapsto \pi_U \circ T \oplus \pi_V \circ T$.
(3) Now let i_U (resp i_V) denote the inclusion $U \to U \oplus V$. Show that these are injective linear transformations and $\pi_V \circ i_U = 0$, $\pi_U \circ i_V = 0$.
(4) Show that we have a natural isomorphism
$$\mathcal{L}(U \oplus V, W) \to \mathcal{L}(U, W) \oplus \mathcal{L}(V, W)$$
given by $T \mapsto T \circ i_U \oplus T \circ i_V$.

3. This problem will play with the notion of "equal" versus "isomorphic". It also stresses some of the pathologies with infinite dimensional spaces. Let $\mathbb{R}[x]$ be the vector space of all polynomials in x over $\mathbb{R}$.

(1) Show that $\mathbb{R}[x]$ is infinite dimensional.
(2) Show that the subspace of all polyomials which have a root at 0 is isomorphic to $\mathbb{R}[x]$. Conclude that if we just know $V \oplus W \cong V \oplus U$, then we can't conclude that $W \cong U$.

(3) Show that the derivative is a surjective map $\mathbb{R}[x] \to \mathbb{R}[x]$. Compute the kernel. Conclude that a surjective map between infinite dimensional vector spaces need not be an isomorphism.
(4) Show that the integral $\int_0^x$ provides a one-sided inverse to the derivative. Determine the image. Conclude that an injective map between infinite dimensional vector spaces need not be an isomorphism.
(5) There is a standard basis for $\mathbb{R}[x]$: the basis of all monic monomials in x. If we mimic the arguments given in class, then we can associate an $\infty \times \infty$-matrix to any linear transformation (indexed by $\mathbb{N} \times \mathbb{N}$, so really it has a well-defined "upper left corner"). Find the matrix for the derivative and integral.
(6) Building on the previous part, does any $\infty \times \infty$-matrix (again, indexed by $\mathbb{N} \times \mathbb{N}$) occur as the matrix associated to a linear operator? What is the image of the map that sends a linear operator to its matrix?