PROBLEM SET #3

DUE DATE: OCTOBER 22

1. Let A be an $n \times n$-matrix, and let B be an $m \times m$-matrix. Let

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in M_{(m+n,m+n)}.$$

For this problem, you have to show that $|C| = |A| \cdot |B|$, using the definition of the determinant.

(1) Let $\Sigma_n \times \Sigma_m$ be

$$\Sigma_n \times \Sigma_m = \big\{ f \in \Sigma_{n+m} | f(\{1, \dots, n\}) = \{1, \dots, n\} \big\}.$$

In other words, we write the set from 1 to $n+m$ as the set from 1 to n then the set from $n+1$ to $n+m$, and we require that f preserves holds these two subsets invariant. Show that if $\sigma \in \Sigma_{n+m} - \Sigma_n \times \Sigma_m$, then

$$c_{1,\sigma(1)} \cdots c_{(n+m),\sigma(n+m)} = 0,$$

where $c_{i,j}$ is the element in position i, j of C. Conclude that

$$|C| = \sum_{\sigma \in \Sigma_n \times \Sigma_m} sign(\sigma) c_{1,\sigma(1)} \cdots c_{(n+m),\sigma(n+m)}.$$

(2) Let $\sigma \in \Sigma_n \times \Sigma_m$, let σ_1 be the restriction of σ to $\{1, \dots, n\}$, and let σ_2 be the restriction of σ to $\{n+1, \dots, n+m\}$. Show that

$$sign(\sigma) = sign(\sigma_1) \cdot sign(\sigma_2).$$

(3) Combine parts 1 and 2 to conclude that

$$|C| = |A| \cdot |B|.$$

2. Let V be a 7 dimensional vector space. Show that there is more than one possible Jordan form for a nilpotent operator on V with index of nilpotency 3 and with a 3-dimensional kernel.

3. Let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$. Let

$$A_p = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}.$$

Show that $p_{A_p}(\lambda) = (-1)^n p(\lambda)$. (Hint: Expand the determinant along the first row or column and use induction on n).

4. Now some examples to work through. Find the Jordan form J of the following matrices A. Find also a Jordan basis and the change of basis matrix P such that $J = P^{-1}AP$.

(1)
$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -2 & 4 & 0 & 1 \\ -4 & 10 & -9 & 6 \end{bmatrix}.$$

(2)
$$B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}.$$

(3)
$$C = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -3 & 6 & -1 & 1 \\ -6 & 13 & -9 & 5 \end{bmatrix}.$$

(4)
$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 \\ 3 & -9 & 9 & -3 & 1 & 0 \\ 7 & -23 & 28 & -16 & 5 & 0 \\ 14 & -49 & 65 & -40 & 10 & 1 \end{bmatrix}.$$