LECTURE 23: METRIC SPACES

This lecture will look at the topological properties that arise from the notion of distance provided by the norm.

Definition 1. If V is an inner product space, then the distance between $\vec{v}$ and $\vec{u}$ is

$$d(\vec{v},\vec{u}) = ||\vec{v}-\vec{u}||.$$

Proposition 1. *The distance function satisfies*

(1) $d(\vec{v},\vec{u}) \geq 0$ *with equality iff* $\vec{v}=\vec{u}$.
(2) $d(\vec{v},\vec{u}) = d(\vec{u},\vec{v})$ *for all* $\vec{u}$ *and* $\vec{v}$
(3) $d(\vec{v},\vec{u}) = d(\vec{v},\vec{w}) + d(\vec{w},\vec{u})$ *for all* $\vec{v}$, $\vec{u}$, $\vec{w}$.

Proof. The first condition is positive-definteness, the second is symmetry, and the third is the triangle inequality. □

Definition 2. A metric space is a set M with a function $d\colon M \times M \to \mathbb{R}$ that satisfies the three conditions in the previous proposition.

Today's lecture will focus on metric spaces and establish some of the basic terminology. Next time we will start to look at the algebraic properties.

First some specific terms.

Definition 3. The open ball of radius r about x is

$$B_r(x) = \{y \in M | d(x,y) < r\}.$$

The closed ball of radius r about x is

$$\overline{B_r(x)} = \{y \in M | d(x,y) \leq r\}.$$

The names come from $\mathbb{R}^2$ with the standard distance. Here

$$B_r(x) = \{(x,y) \in \mathbb{R}^2 | d\big((a,b),(x,y)\big) < r\} = \{(x,y) | (x-a)^2 + (y-b)^2 < r^2\}.$$

We can, however, put other distances on $\mathbb{R}^n$.

Definition 4. A <u>norm</u> on V is a function $||-||\colon V \to \mathbb{R}$ such that

(1) $||\vec{v}|| \geq 0$ with equality iff $\vec{v} = \vec{0}$.
(2) $||r\vec{v}|| = |r| \cdot ||\vec{v}||$
(3) $||\vec{v}+\vec{u}|| \leq ||\vec{v}|| + ||\vec{u}||$.

There are a number of important examples.

(1) Let $p \geq 1$, and let $||-||_p$ on $\mathbb{R}^n$ be

$$||\vec{v}||_p = \sqrt[p]{\sum |a_i|^p}.$$

If $p=2$, then this is our Euclidean distance. If $p=1$, then this is the so called "taxi-cab" distance, since the distance between two points on the plane is how far a taxi would drive if it could only travel parallel to the x or y axis. For other values of p, we don't have such a fun description.

The unit ball centered at the origin in each of these looks very different depending on the value of p. If $p = 1$, then it looks like a diamond with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. If $p = 2$, then it is the standard unit circle, and as $p \to \infty$, it grows to the square with vertices at $(\pm 1, \pm 1)$.

(2) Let $|| - ||_\infty$ on $\mathbb{R}^n$ be
$$||\vec{v}||_\infty = \max_{i \leq n} |a_i|.$$
This serves as a kind of limit of the $|| - ||_p$ (if I hold $\vec{v}$ constant and let $p \to \infty$, then the norms converge to the infinity norm).

The unit ball in this metric space is the square with vertices at $(\pm 1, \pm 1)$.

(3) ℓ_p: these are all infinite sequences of real numbers such that the p-norm makes sense (in other words, such that the associated sequence converges). For the infinity norm, these are all bounded sequences.

(4) $L^p([a,b])$: these are all functions $[a,b] \to \mathbb{R}$ such that $\int_a^b |f(x)|^p dx < \infty$. The norm is then the p^{th} root of this integral. In particular, all continuous functions sit inside this space, but there are also discontinuous functions here too.

The reason we consider metric spaces is that we can use this to talk about convergences of sequences.

Definition 5. Let (x_n) be a sequence of elements in a metric space M. We say that (x_n) converges to x if for every $\epsilon > 0$, we can find an N such that for all $n > N$, $x_n \in B_\epsilon(x)$.

Put another way, if you give me some distance from x (in this case, ϵ), then all of the terms in the sequence are eventually within ϵ of x.

The fact that for all distinct x, y, $d(x,y) > 0$ shows us that if a limit exists, then it must be unique (we just choose ϵ to be less than $\frac{1}{2}d(x,y)$ and use the triangle inequality to see that we can't be within epsilon of x and of y).

In our case, we will use limits to talk about infinite series (as the limits of the partial sums). We'll focus on this in a later lecture. For now, we continue with just topological properties.

Definition 6. A set U is open if for every $x \in U$, there is an r such that $B_r(x) \subset U$.

A set T is closed if T^c is open.

The following is immediate from the definition.

Proposition 2. (1) *$\emptyset$ and M are open.*
(2) *If U and V are open, then so is $U \cap V$.*
(3) *If U_i, $i \in I$ is a collection of open sets, then $\bigcup U_i$ is open.*

A restatement of our result about limits is that x is the limit of (x_n) iff for every open set U about x, all but finitely many of the terms in the sequence are in U.

A corresponding result is true for the closed sets, and we will return to that later. Our focus now is on limits.

Definition 7. A sequence (x_n) is Cauchy if for all $\epsilon > 0$, there exists an N such that for all $n, m > N$,
$$d(x_n, x_m) < \epsilon.$$

In other words, a sequence is Cauchy if the terms get arbitrarily close together as we proceed in the sequence.

Proposition 3. *If (x_n) converges to x, then (x_n) is Cauchy.*

Proof. We apply the triangle inequality to $d(x_n, x_m)$:

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m).$$

Now choose N such that $d(x_n, x) < \epsilon/2$ for all $n > N$, and we have that (x_n) is Cauchy. □

The converse is not in general true. Consider the metric space of real numbers with the absolute value metric. The subspace of rational numbers is a submetric space, and in this, not every Cauchy sequence converges. In particular, take the decimal approximations to your favorite irrational number. This is a Cauchy sequence since it converges in $\mathbb{R}$, but not in $\mathbb{Q}$.

Definition 8. A metric space M is complete if every Cauchy sequence converges.

Like everything else we have talked about, there is a secret universal property underlying the following theorem. The proof of this is somewhat involved, so we will give only a sketch of the key ideas.

Theorem 1. *Every metric space can be embedded in a complete metric space.*

In fact, we can always embed M as a dense subset, but such a discussion will have to await our discussion of density. We also postpone slightly what we mean by "embed". For now, we just describe the completion.

Let $\tilde{M}$ denote the collection of all Cauchy sequences from M. We can embed M in here as the constant sequences. We then have that a Cauchy sequence in M converges to ... that Cauchy sequence as an element of $\tilde{M}$. We can define a pseudometric on $\tilde{M}$ by saying

$$d\big((x_n), (y_n)\big) = \lim d(x_n, y_n).$$

The problem here (and the reason for the "pseudo-") is that different Cauchy sequences can converge to the same element (for example, in $\mathbb{R}$, we have, say, decimal and binary expansions for an irrational number). This means that our distance function is not always non-negative for distinct inputs. We can fix this by declaring that two elements are equivalent if $d(x, y) = 0$. The set of equivalence classes inherits a distance from d, and this is a metric space. We let $\bar{M}$ denote the quotient of $\tilde{M}$ by this equivalence relation.

It is not difficult (but somewhat tedious) to show that $\bar{M}$ is a complete metric space. It is not difficult to show that M embeds therein, completing the argument.

As an aside, this is one of the two main constructions of the real numbers. We begin with the rationals as a metric space and we complete them to the reals by adding in the limits of Cauchy sequences that don't converge in $\mathbb{Q}$ (namely the irrational numbers).

To finish the lecture, we turn to a notion of continuous. This was underlying our notion of "embedding" (though there we will use a slightly stronger notion of "isometry"). Recall the definition from one variable calculus:

Definition 9. A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at a if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

We can copy this definition *mutatis mutandis* to the metric space context: we replace $|x - a| < \delta$ with $x \in B_\delta(a)$ and $|f(x) - f(a)| < \epsilon$ with $f(x) \in B_\epsilon(f(a))$. We can do better, and this is often easier to check.

Proposition 4. *A function $f: M \to N$ between two metric spaces is continuous if for all open sets U in N, $f^{-1}(U)$ is open in M.*

This is an elementary exercise in the definitions. This relationship provides the *definition* of continuous for functions between arbitrary topological spaces: the inverse image of open sets is open.

We close with an interpretation of this: open sets measure “closeness”. This proposition / definition says “close things map to close things”, which is exactly our intuition of continuity.