LECTURE 20: APPLICATIONS OF EXTERIOR PRODUCTS

Today will be a little different. Rather than focus too much on theory, we will look at two very interesting examples involving the exterior product. But before then, we have to talk a little more about "functoriality". I'll post some notes about category theory later to make it more precise.

Our construction of $\Lambda^k(V)$ is "functorial": given $T\colon V \to W$, we have a linear map

$$\Lambda^k(T)\colon \Lambda^k(V) \to \Lambda^k(W),$$

and we have two properties:

(1) $\Lambda^k(Id) = Id$
(2) $\Lambda^k(T) \circ \Lambda^k(S) = \Lambda^k(T \circ S)$.

So how does this work? Given $\vec{v}_1 \wedge \cdots \wedge \vec{v}_k \in \Lambda^k(V)$, we define $\Lambda^k(T)$ as

$$\Lambda^k(T)(\vec{v}_1 \wedge \cdots \wedge \vec{v}_k) = T(\vec{v}_1) \wedge \cdots \wedge T(\vec{v}_k).$$

Let's work out an example. We'll be letting $V = \mathbb{R}^3$.

Example. *Let*

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We talked last time about a basis for $\Lambda^2(\mathbb{R}^3)$:

$$\{\vec{e}_1 \wedge \vec{e}_2, \vec{e}_1 \wedge \vec{e}_3, \vec{e}_2 \wedge \vec{e}_3\}.$$

We'll use this as an ordered basis. Now we'll compute $\Lambda^2(T)$.

$$\Lambda^2(T)(\vec{e}_1 \wedge \vec{e}_2) = T(\vec{e}_1) \wedge T(\vec{e}_2) = (\vec{e}_1 + \vec{e}_2) \wedge (\vec{e}_2 + \vec{e}_3) =$$
$$\vec{e}_1 \wedge \vec{e}_2 + \vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_2 + \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2 + \vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_3.$$

We can do this for the other two basis vectors, and we find

$$\Lambda^2(T) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Determinants

Now let $V = W$, so $T\colon V \to W$ is an endomorphism $T\colon V \to V$. Let's assume $\dim(V) = n$. If we consider the top exterior power, then we have a map

$$\Lambda^n(T)\colon \Lambda^n(V) \to \Lambda^n(V).$$

Since $\dim V = n$, we saw that $\dim \Lambda^n(V) = 1$. Now we make a slightly unnatural choice: choose a basis for V. This gives us a basis for $\Lambda^n(V)$, and therefore an isomorphism with $\mathbb{F}$. The ground field $\mathbb{F}$ has a canonical basis element 1. Given a linear map $S\colon \mathbb{F} \to \mathbb{F}$, we can evaluate on 1 and we get an element of $\mathbb{F}$. This establishes that all linear maps $\mathbb{F} \to \mathbb{F}$ are given by multiplication by an element of $\mathbb{F}$ $(S(1))$.

Now multiplication by an element is independent of choice of basis, thus though we used a basis early on, the answer doesn't in any way depend on the basis. Putting this together, we learn that $\Lambda^n(T)$ is multiplication by a number. We have to compute that number.

Proposition. *The linear map $\Lambda^n(T)$ is multiplication by $|[T]_{\mathcal{B}}|$ for any basis $\mathcal{B}$.*

The proof of this will actually be a homework exercise (after one choses a basis). Instead we'll now do an illustrative example, using our linear map T on $\mathbb{R}^3$ from before.

Example. *We compute $\Lambda^3(T)$.*

$$T(\vec{e}_1) \wedge T(\vec{e}_2) \wedge T(\vec{e}_3) = (\vec{e}_1 + \vec{e}_2) \wedge (\vec{e}_2 + \vec{e}_3) \wedge (\vec{e}_1 + \vec{e}_3) = \\ \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_1 = 2\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3.$$

Thus $\Lambda^3(T)$ is multiplication by 2. What went into this?

1. *If a term was repeated, then it was killed in the wedge (so we never saw $\vec{e}_i \wedge \vec{e}_i$). This has the effect of saying we never see two entries from the same column in a matrix for T in the same term.*
2. *We needed a factor of $(-1)^{\text{number of terms out of order}}$ to switch back to the standard order.*

Corollary. *The determinant is multiplicative:*

$$|A \cdot B| = |A||B|.$$

Proof. A and B both correspond to linear operators T and S. Functoriality shows

$$\Lambda^n(T \circ S) = \Lambda^n(T) \circ \Lambda^n(S).$$

The left-hand side is multiplication by $|A \cdot B|$. The right-hand side is multiplication by $|B|$ followed by multiplication by $|A|$, so multiplication by $|A||B|$. □

This proof is much easier than the standard one. There was no manipulation of indices, etc, involved!

Corollary. *The determinant of T is independent of choice of basis, and hence, the determinant is invariant under conjugation.*

We already saw this: the choice of basis didn't matter (since multiplication is independent of basis).

Thus we have a very clean, natural description of the determinant that works for any linear operator. Fun!

The Cross Product

Our second example is related to the first, and it will provide some geometric intuition. We'll restrict now to $V = \mathbb{R}^3$, and we'll call the basis vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$. Our work from last time showed that the basis $\vec{\imath} \wedge \vec{\jmath} \wedge \vec{k}$ for $\Lambda^3(\mathbb{R}^3)$ gave us a perfect pairing (and hence an isomorphism)

$$\Lambda^2(\mathbb{R}^3) \to (\mathbb{R}^3)^*.$$

We'll evaluate that now. Recall that the map is

$$(\vec{v} \wedge \vec{w}) \mapsto f_{\vec{v} \wedge \vec{w}} := (\vec{u} \mapsto \vec{v} \wedge \vec{w} \wedge \vec{u}).$$

Of course, we need only determine everything for basis vectors.

$$f_{\vec{\imath}\wedge\vec{\jmath}}(a\vec{\imath} + b\vec{\jmath} + c\vec{k}) = c\vec{\imath} \wedge \vec{\jmath} \wedge \vec{k}.$$

Thus we conclude that $f_{\vec{\imath}\wedge\vec{\jmath}} = \delta_{\vec{k}}$, the dual basis vector for $\vec{k}$. Identical computations gives the following

$$\vec{\imath} \wedge \vec{\jmath} \mapsto \delta_{\vec{k}}, \quad \vec{\imath} \wedge \vec{k} \mapsto -\delta_{\vec{\jmath}}, \quad \vec{\jmath} \wedge \vec{k} \mapsto \delta_{\vec{\imath}}.$$

We've already chosen a basis, so we can follow this map with the identification of a vector space with its dual using that. Thus $\delta_{\vec{v}} \mapsto \vec{v}$. Now we have a linear map

$$\Lambda^2(\mathbb{R}^3) \to \mathbb{R}^3$$

and under this $\vec{\imath} \wedge \vec{\jmath} \mapsto \vec{k}$, $\vec{\imath} \wedge \vec{k} \mapsto -\vec{\jmath}$, and $\vec{\jmath} \wedge \vec{k} \mapsto \vec{\imath}$.

This *is* the cross product. Recall that the cross product is a map

$$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

that is linear in each factor (hence bilinear) and skew-symmetric (hence alternating, since we are over $\mathbb{R}$). That means it's the same thing as a linear map $\Lambda^2(\mathbb{R}^3) \to \mathbb{R}^3$, and we see that in fact, it's the most natural possible linear map!

So what's the difference between the right-hand rule (what we used) and the left-hand rule? $\mathbb{R}^3$ has a very natural automorphism: multiplication by -1 (let's call this map M. The essential difference between the two rules is the choice of the one we did, or -1 times it.

Now we see again why the formula we learned in multivariable calculus is so horrible. We are looking at the exterior square of something and linking it back to the original space. This works EXACTLY because ... $2 = 3 - 1$. For spaces of higher dimension, there isn't a bilinear cross product; there will be a multilinear one.

Now we can bring in more geometry to explain what the determinant is trying to tell us. Remember from Calc III that the volume of a parallelepiped with edges from a single vertex $\vec{v}$, $\vec{w}$, $\vec{u}$. The volume is given by $\vec{v} \cdot (\vec{w} \times \vec{u})$. We can now recast this in the following way. Given three linearly independent vectors (in other words, the edges of a non-degenerate parallelepiped), we have a linear transformation which sends each of the standard basis vectors to the corresponding edge vector. The determinant of this linear transformation is $\vec{v} \cdot (\vec{w} \times \vec{u})$. That tells us what the more general form of the determinant is: the top exterior power is multiplication by the determinant and for $\mathbb{R}^3$ (or more generally for $\mathbb{R}^n$), this number is the factor by which volumes scale.

This is also exactly what shows up for coordinate changes. We should think of the coordinate changes as a smooth bijection from $\mathbb{R}^n$ to itself. In other words, each of the new coordinates can be realized as smooth functions of the old ones.

Now a small review of multivariable calculus. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. Given a point $p \in \mathbb{R}^n$, let $T_p(\mathbb{R}^n)$ be the tangent space to $\mathbb{R}^n$ at p. For this, you consider all paths $\gamma\colon (-1, 1) \to \mathbb{R}^n$ such that $\gamma(0) = p$. Then $\gamma'(0)$ is a tangent vector at p, and all such vectors are of this form. This gives us a copy of $\mathbb{R}^n$ (since we just consider the lines through p in the coordinate directions). Given $\gamma'(0)$, we can get a tangent vector at $f(p)$ by considering $(f \circ \gamma)'(0)$. The usual rules show us that this is $f'(\gamma(0)) \cdot \gamma'(0)$. However, $f'(\gamma(0))$ is now an $m \times n$-matrix. It is a linear map that takes the tangent vectors at $T_p(\mathbb{R}^n)$ to $T_{f(p)}(\mathbb{R}^m)$. If we write

f as

$$f(x_1, \ldots, x_n) = (y_1, \ldots, y_m),$$

where each y_i is a function of the variables x_1 to x_n. Then the matrix $f'(p)$ is given by

$$f'(p) = \left[\frac{\partial y_i}{\partial x_j}\right].$$

In the change of coordinates examples, the determinant of the derivative f' measures exactly the "local" change in volume produced by the change of coordinates map. We call this determinant the "Jacobian", and it serves as the "change of coordiantes" for multiple integrals. Another classical concept made clearer!