LECTURE 19: EXTERIOR PRODUCTS II

We ended last time looking at a basis for the exterior powers. Today we will finish with that and describe the symmetric power. First we want a generalization of the Reisz representation theorem.

Proposition. *If $f\colon U \otimes V \to W$ is bilinear, then we have canonical maps $U \to \mathcal{L}(V, W)$ and $V \to L(U, W)$ given by*

$$\vec{u} \mapsto f(\vec{u}, -) \text{ and } \vec{v} \mapsto f(-, \vec{v}).$$

Now if W is 1-dimensional, then $\mathcal{L}(V, W) \cong V^*$. Additionally, if for all $0 \neq \vec{u} \in U$, we have a $\vec{v} \in V$ such that $f(\vec{u}, \vec{v}) \neq 0$, then our map is actually an injection $U \to V^*$. Similarly, if for every $0 \neq \vec{v} \in V$ we have a $\vec{u} \in U$ such that $f(\vec{u}, \vec{v}) \neq 0$, then $V \to U^*$ is an injection. Thus if both conditions are satisfied, and if everything in sight is finite dimensional, then our bilinear function f establishes isomorphisms $U \to V^*$ and $V \to U^*$. This is the case with exterior products, as we shall soon see.

If f establishes an isomorphism $U \to V*$ and $V \to U^*$, then we say that f is a perfect pairing.

Let's assume for now that $\Lambda^m(V)$ is one dimensional. Our earlier argument shows that it is at most 1-dimensional. In fact, if we know that this is 1-dimensional, then we can prove that the listed elements are all linearly independent, as we'll soon see.

Proposition. *The canonical map*

$$\Lambda^i(V) \otimes \Lambda^{n-i}(V) \to \Lambda^n(V)$$

given by $\vec{v} \otimes \vec{w} \mapsto \vec{v} \wedge \vec{w}$, is a perfect paring.

Before continuing, we spend a little time describing the map. Let I_i denote the subspace of $V^{\otimes i}$ spanned by vectors of the form $\vec{v}_1 \otimes \dots \vec{v}_i \otimes \dots \vec{v}_i \otimes \dots \vec{v}_n$. We'll use the canonical identification $V^{\otimes i} \otimes V^{\otimes(n-i)}$ with $V^{\otimes n}$ and henceforth blur the distinction between the two. This gives us subspaces $I_i \otimes V^{\otimes(n-i)}$ and $V^{\otimes i} \otimes I_{n-i}$ of $V^{\otimes n}$. It is immediate that

$$\Lambda^i(V) \otimes \Lambda^{n-i}(V) \cong V^{\otimes n}/(I_i \otimes V^{\otimes(n-i)} + V^{\otimes i} \otimes I_{n-i}),$$

and the map realizing this is the tensor product of the two projection maps

$$V^{\otimes i} \to \Lambda^i(V) \text{ and } V^{\otimes(n-i)} \to \Lambda^{n-i}(V).$$

Since $(I_i \otimes V^{\otimes(n-i)} + V^{\otimes i} \otimes I_{n-i})$ is visibly a subspace of I_n, we conclude from the defining property of the quotient that there is a map

$$\Lambda^i(V) \otimes \Lambda^{n-i}(V) \to \Lambda^n(V).$$

This is our map above.

We can spell out a little more what this means and why we would expect such a thing. The object $\Lambda^i(V)$ represents "alternating, multilinear maps from V^i":

$$\mathcal{L}(\Lambda^i(V), W) = \{f\colon V^{\times i} \to W | f \text{ is alternating, multilinear}\}.$$

Tensor product has a similar property. Thus $\mathcal{L}(\Lambda^i(V) \otimes \Lambda^{n-i}(V), W)$ is the set of multilinear maps $V^{\times i} \times V^{\times(n-i)} \cong V^{\times n}$ to W that are alternating in the first i variables and alternating in the last $(n-i)$-variables. On the other hand, if we are given a function $V^{\times n} \to W$ that is multilinear and alternating in all variables, then in particular, it is multilinear and alternating in the first i and last $(n-i)$ variables. Thus we see that for all W,

$$\mathcal{L}(\Lambda^n(V), W) \subset \mathcal{L}(\Lambda^i(V) \otimes \Lambda^{n-i}(V), W).$$

We actually know more. This is "natural" in W. If you give a map $W \to W'$, then composing with this map gives a natural square with inclusions. That alone tells us for purely formal reasons that we have a linear map

$$\Lambda^i(V) \otimes \Lambda^{n-i}(V) \to \Lambda^n(V),$$

precomposition by which is realizes the inclusions (and in fact, the "inclusions" part guarantees that this map is a surjection). Where does it come from? If we let $W = \Lambda^n(V)$, then we have the identity map in

$$\mathcal{L}\big(\Lambda^n(V), \Lambda^n(V)\big) \subset \mathcal{L}\big(\Lambda^i(V) \otimes \Lambda^{n-i}(V), \Lambda^n(V)\big).$$

The image of the identity map is again our map.

Now back to the proof.

Proof. We must show that for each $\vec{v} \in \Lambda^i(V)$, there is a $\vec{v}' \in \Lambda^{n-i}(V)$ such that $\vec{v} \wedge \vec{v}' \neq 0$. Interchanging the roles of i and $n-i$ will then show the other direction too. We start with our spanning set, and make a few minidefinitions.

A small bit of notation. Let $I \subset [n] = \{1, \dots, n\}$ be a set of cardinality i, and let I^c denote the complement. For a subset J of $[n]$, let

$$\vec{v}_J = \bigwedge_{j \in J} \vec{v}_j.$$

Then the following is an easy exercise in alternatingness: If $|I| + |J| = n$, then

$$\vec{v}_I \wedge \vec{v}_J = \begin{cases} 0 & J \neq I^c \\ \pm \vec{v}_M & J = I^c. \end{cases}$$

Now since $\vec{v}_I$ spans where I ranges over the subsets of cardinality i, we know that any $\vec{v}$ can be written as a linear combination

$$\vec{v} = a_{I_1} \vec{v}_{I_1} + \cdots + a_{I_m} \vec{v}_{I_m},$$

where without loss of generality, we may assume $a_{I_1} \neq 0$. Then by the previous result, if we let $\vec{v}' = \vec{v}_{I_1^c}$, then

$$\vec{v} \wedge \vec{v}' = \pm a_{I_1} \vec{v}_{[n]} \neq 0,$$

as required. $\square$

As an easy corollary, we conclude that our spanning sets are also linearly independent.

Proposition. *If we assume* $\dim \Lambda^m(V) = 1$*, then*

$$\{\vec{v}_{i_1} \wedge \cdots \wedge \vec{v}_{i_n} | i_1 < \cdots < i_n\}$$

is a basis for $\Lambda^n(V)$*.*

Proof. Now consider a linear dependence relation

$$a_{I_1}\vec{v}_{I_1} + \cdots + a_{I_k}\vec{v}_{I_k} = 0.$$

If we wedge this with $\vec{v}_{I_1^c}$, then the above observation shows us that all terms except the first drop out. Now since $\vec{v}_M$ spans $\Lambda^m(V)$ and since this is a 1-dimensional space, we know $\vec{v}_M \neq \vec{0}$. Thus our dependence relation becomes

$$a_{I_1}\vec{v}_M = 0,$$

which means $a_{I_1} = 0$. Induction on k gives the result. □

As a consequence, we see that wedge product map $\Lambda^n(V) \otimes \Lambda^{m-n}(V) \to \Lambda^m(V)$ establishes an isomorphism between $\Lambda^m(V)$ and $\Lambda^{n-m}(V)$ as described above. In fact, we have an even better result: the basis dual to $\{\vec{v}_I\}$ is the basis $\{\vec{v}_{I^c}\}$. Thus we recover a very classical result:

$$\binom{m}{n} = \dim \Lambda^n(V) = \dim \Lambda^{m-n}(V) = \binom{m}{m-n}.$$

To complete the proof in the exterior case, we have to produce a single non-zero, alternating, multilinear function on m-tuples of vectors in V. This will show that $\Lambda^m(V) \neq 0$, since if it were zero, we'd know that all such functions are zero (as they factor through the zero space). V is an m-dimensional space, so our choice of basis identifies this with $\mathbb{F}^m$. Now we will be done if we can find a single non-zero alternating multilinear function on m-tuples on column vectors. Of course, an m-tuple of m-dimensional column vectors is the same thing as an $m \times m$-matrix, and this gives us a great example.

Proposition. *The determinant, viewed as a function from m-tuples of m-dimensional column vectors, is alternating, multilinear, and non-zero.*

Corollary. *The vector space $\Lambda^m(V)$ is one dimensional.*