LECTURE 18: SYMMETRIC AND EXTERIOR PRODUCTS

We established in the last two lectures ways to identify bilinear maps from $V \times W$ to U with linear maps $V \otimes W$to U and ways to better understand the construction of the tensor product. If we have extra properties of our bilinear map, then we can push these through to the tensor product. In this class, we will focus on those cases where $V = W$, so our bilinear maps are direct generalizations of bilinear forms. We therefore use the same language.

Definition. *A bilinear map $f \colon V \times V \to W$ is...*

<u>symmetric</u> if $f(\vec{v}, \vec{w}) = f(\vec{w}, \vec{v})$
<u>skew-symmetric</u> if $f(\vec{v}, \vec{w}) = -f(\vec{w}, \vec{v})$
<u>alternating</u> if $f(\vec{v}, \vec{v}) = 0$.

We can then look at all symmetric, skew-symmetric, or alternating functions on $V \times V$. Rewriting our expressions which define these properties, we learn that these forms are close related to particular families of elements in $V \otimes V$ which are automatically in the kernel.

Proposition. *Let $F \colon V \otimes V \to W$ be a linear map corresponding to a bilinear function f (which makes F essentially f). Then*

f is symmetric if and only if $\vec{v} \otimes \vec{w} - \vec{w} \otimes \vec{v} \in \ker F$
f is skew-symmetric if and only if $\vec{v} \otimes \vec{w} + \vec{w} \otimes \vec{v} \in \ker F$
f is alternating if and only if $\vec{v} \otimes \vec{v} \in \ker F$.

In fact, this proposition is obvious. We simply recall that $F(\vec{v} \otimes \vec{w}) = f(\vec{v}, \vec{w})$, so the proposition is immediate.

Just as with bilinear forms, alternating implies skew-symmetric (and conversely in characteristic not 2), so we will only focus on symmetric and alternating. Since the two things are somewhat unrelated, we will also focus today on alternating forms. Next time, we will talk about symmetric forms.

Just as with the tensor product, we have a universal vector space that realizes all of these kinds of structures.

Definition. *Let V be a vector space. The exterior square of V, $\Lambda^2(V)$, is the quotient of the tensor square by the subspace generated by $\vec{v} \otimes \vec{v}$.*

Proposition. *For any W, $\mathcal{L}(\Lambda^2(V), W)$ is the same as the set of alternating bilinear functions $V \times V \to W$.*

Proof. We saw in the above proposition that any element of the form $\vec{v} \otimes \vec{v}$ is in the kernel of the linear transformation associated to an alternating bilinear function. Thus by the universal property of the quotient, this linear transformation factors through the quotient space, $\Lambda^2(V)$. Conversely, if L is a linear transformation from $\Lambda^2(V) \to W$, then composing with the canonical projecting $V \otimes V \to \Lambda^2(V)$ gives a linear transformation from $V \otimes V$ to W, and thus a bilinear function. Since the necessary elements are in the kernel, we learn that this is an alternating bilinear function. These two identifications are clearly inverses to each other. $\square$

Some notation is in order. For the exterior product, we normally replace the symbol $\otimes$ between vectors with $\wedge$. Thus elements in $\Lambda^2(V)$ are linear combinations of the form $\vec{v}\wedge\vec{w}$, and we remember that $\vec{v}\wedge\vec{w} = -\vec{w}\wedge\vec{v}$. There are two important and beautiful things which arise from this circle of ideas, but first we must slightly generalize to an arbitrary number of variables.

Definition. *A multilinear function $f\colon U\times\cdots\times U\to W$ (with n copies of U) is* alternating *if $f(\vec{u}_1,\ldots,\vec{u}_n)=0$ whenever $\vec{u}_i=\vec{u}_j$ for some $i\neq j$.*

Just as before, we an obvious relationship between these and elements in the n^{th} tensor power. This lets us make the following definitions.

Definition. *The n^{th} exterior power of V, $\Lambda^n(V)$ is the quotient of $V^{\otimes n}$ by the subspace generated by all elements of the form*

$$\vec{u}_1\otimes\cdots\otimes\vec{u}_j\otimes\cdots\otimes\vec{u}_i\otimes\cdots\otimes\vec{u}_n$$

where $\vec{u}_i=\vec{u}_j$.

Just as before, we normally replace $\otimes$ in $\Lambda^n(V)$ with $\wedge$. In $S^n(V)$, we have forms for which the order doesn't matter at all. In $\Lambda^n(V)$, if we swap any two vectors then the sign changes.

What do these spaces look like? In particular, if we have an ordered basis for V, say $\{\vec{v}_1,\ldots,\vec{v}_m\}$, how do we produce a basis for $\Lambda^n(V)$.

Theorem. *The set*

$$\{\vec{v}_{i_1}\wedge\cdots\wedge\vec{v}_{i_n}|i_1<\cdots<i_n\}$$

forms a basis for $\Lambda^n(V)$.

Proof. We will show that this set spans. We note that as before, the set

$$\{\vec{v}_{i_1}\otimes\cdots\otimes\vec{v}_{i_n}\}$$

with no restrictions on the subscripts forms a basis for $V^{\otimes n}$. Since alternating implies skew-symmetric, we know that any element $\vec{v}_{i_1}\otimes\ldots\vec{v_{i_n}}$ maps to the same element (possibly with a sign) in $\Lambda^n(V)$ as the element with the indices in non-decreasing order. Thus

$$\vec{v}_{i_1}\wedge\cdots\wedge\vec{v}_{i_n}$$

with $i_1\leq\cdots\leq i_n$ span $\Lambda^n(V)$. However, if any index is repeated, then we know that vector is zero. Thus we can restrict to those with a strictly increasing sequence of indices.

Linear independence is MUCH trickier. We will postpone the argument for a time. $\square$

Assuming linear independence, then we have the following consequence.

Corollary. *If V has dimension m, then we have*

(1) $\dim V^{\otimes n}=m^n$
(2) $\dim\Lambda^n(V)=\binom{m}{n}$,

where $\binom{i}{j}=\frac{i!}{(i-j)!j!}$ is the binomial coefficient.

This is a simple counting argument. You will prove these statements as an exercise.