Lecture 12 - Jordan Form

Saw last time that if $\{\bar{v}_1, \dots, \bar{v}_r\} \subseteq \ker(A^k)$ such that $\{A^{k-1}\bar{v}_1, \dots, A^{k-1}\bar{v}_r\}$ is linearly independent, then $\{\bar{v}_1, \dots, \bar{v}_r, A\bar{v}_1, \dots, A^{k-1}\bar{v}_r\}$ is linearly independent. Let's reorder:

$$\{\bar{v}_1, A\bar{v}_1, \dots, A^{k-1}\bar{v}_1\} \cup \dots \cup \{\bar{v}_r, \dots, A^{k-1}\bar{v}_r\}$$

__Def__ A cyclic subspace for A is one of the form

$$\langle \bar{v}, A\bar{v}, A^2\bar{v}, \dots \rangle = C_A(\bar{v})$$

So in the set-up of the previous theorem, if $\{A^{k-1}\bar{v}_1, \dots\}$ is linearly ind, then $\langle \bar{v}_1, \dots, A^{k-1}\bar{v}_r \rangle$ is a direct sum of r cyclic subspaces

$$C_A(\bar{v}_1) \oplus \dots \oplus C_A(\bar{v}_r)$$

__Prop__ With respect to the basis $\{A^{k-1}\bar{v}_1, \dots, \bar{v}_1\}$, $A|_{C_A(\bar{v}_1)}$ becomes $J_k(0)$.

__Pf__ This is the computation from last time.

The point of Jordan form is that while not every vector space splits as a sum of eigenspaces, every generalized eigenspace splits as a sum of cyclic subspaces.

__Thm__ (Jordan Form) If $A^k = 0$, then

$$V = C_A(\bar{v}_1) \oplus \dots \oplus C_A(\bar{v}_r) \quad \text{for some } \bar{v}_1, \dots, \bar{v}_r \in V.$$

What does this mean for us?

__Cor__ Let $l_i = \dim C_A(\bar{v}_i)$. Then

$$A \simeq \begin{bmatrix} J_{\ell_1}(0) & & \bigcirc \\ & J_{\ell_2}(0) & \\ \bigcirc & \bigcirc \quad \ddots & J_{\ell_r}(0) \end{bmatrix}$$

Pf: The decomp into a sum of invariant cyclic spaces ensures we have this block form. The blocks are given by A restricted to each factor, and this is the previous proposition. ☒

Pf of Theorem: By induction on k = index of nilpotency on V. If $k=1$, it's obvious. So assume true for any index $< k$.

Choose a basis $\{x_1, \dots, x_m\}$ for $\ker(A^{k-1})$ and extend to a basis $\{x_1, \dots, x_m, \bar{v}_1, \dots, \bar{v}_r\}$ for V. Then $\{A^{k-1}\bar{v}_1, \dots, A^{k-1}\bar{v}_r\}$ is linearly independent:

$$a_1 A^{k-1}\bar{v}_1 + \dots + a_r A^{k-1}\bar{v}_r = \bar{0}$$
$$\Updownarrow$$
$$A^{k-1}(a_1\bar{v}_1 + \dots + a_r\bar{v}_r) = \bar{0}$$
$$\Updownarrow$$
$$a_1\bar{v}_1 + \dots + a_r\bar{v}_r \in \ker(A^{k-1}) = \langle \bar{x}_1, \dots, \bar{x}_m \rangle$$
$$\Rightarrow a_1 = \dots = a_r = 0.$$

So by our earlier theorem, $\{\bar{v}_1, \dots, \bar{v}_r, \dots A^{k-1}\bar{v}_r\}$ is linearly independent. Now note that by construction, any $\bar{v} \in V$ can be written as $\bar{v}_1 + \bar{v}_0$, where $\bar{v}_1 \in \langle \bar{v}_1, \dots, \bar{v}_r \rangle$ and $\bar{v}_0 \in \ker(L^{k-1})$. Here our inductive step will kick in with a caveat:

When we extend our basis for $\ker(A^{k-2})$ to one of $\ker(A^{k-1})$, we include our chosen vectors (in this case, $A\bar{v}_1, \dots, A\bar{v}_r$). Then by the inductive hypothesis, we are done. ☒

Now several lengthy examples.

Example 1

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

Step 1 Find $p_A(\lambda)$ and eigenvalues

$$p_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 2 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -\lambda \\ 1 & -1 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 - 2\lambda + 2) + (\lambda - 1)$$

$$= (1-\lambda)^3$$

eigenvalues: 1 w/ multiplicity 3

Step 2 Determine the smallest k s.t. $\ker((A-\lambda I)^k) = \ker((A-\lambda I)^{k+1})$

It's conveient to write $B = A - I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$$B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\ker(B^2) \subsetneq \ker(B^3) = \mathbb{F}^3$, and $k = 3$.

Step 3 Extend a basis for $\ker(B^2)$ to one of $\ker(B^3)$.

$\ker(B^2) = \left\langle \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\rangle$, so a basis is $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$

$\ker(B^3) = \mathbb{F}^3$, so just need one linearly independent vector: $\bar{e}_1$.

Step 4 Take the cyclic subspace generated by the new vectors.

$$\bar{e}_1, \quad B\bar{e}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad B^2\bar{e}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Step 5 See if this spans $\ker(B^k)$; if not, repeat

These span.

Jordan basis:

$$\bar{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Since $B = A - I$,

$$B\bar{v}_1 = \bar{0} \Rightarrow A\bar{v}_1 = \bar{v}_1$$

$$B\bar{v}_2 = \bar{v}_1 \Rightarrow A\bar{v}_2 = \bar{v}_2 + \bar{v}_1$$

$$B\bar{v}_3 = \bar{v}_2 \Rightarrow A\bar{v}_3 = \bar{v}_3 + \bar{v}_2$$

So in the $\bar{v}$-basis, A becomes

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}; \quad \text{If } P = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 0\end{bmatrix}, \text{ then}$$

$$A = PJ_3(1)P^{-1}.$$

Example 2

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix}$$

Step 1 $p_A(\lambda)$ and eigenvalues

$$\begin{vmatrix} -1-\lambda & 1 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ -2 & 2 & \lambda & 0 \\ 0 & 3 & -1 & 1-\lambda \end{vmatrix} = (\lambda-1)\cdot\lambda^3$$

$\lambda = 1, 0$ (mult 3)

Repeat steps 2-5 for each eigenvalue

$\lambda = 1$:

Step 2: Smallest k w/ $\ker((A-I)^k) = \ker((A-I)^{k+1})$

Since 1 has multiplicity 1, $k = 1$ (k is always $\leq$ multiplicity)

Step 3: Find a basis for $\ker(A-I)$

$\{\bar{e}_4\}$ works.

Step 4: Take the cyclic subspace gen by $\{\bar{e}_4\}$

This is $\langle \bar{e}_4 \rangle$

Step 5: Compare w/ $\ker(A-I)$

This is $\ker(A-I)$.

$\lambda = 0$

Step 2:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 1 \end{bmatrix} = A^2$$

So $\ker(A^2) = \ker(A^3)$

Step 3

$$\ker(A) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

$$\ker(A^2) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$$

↑ new vector

Step 4:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \end{bmatrix}$$

Step 5:

Missing one dim in $\ker(A)$, so add in $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Jordan Basis:

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

; in this basis, we get

$$\left[\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$