Lecture 11 - Jordan Form

Last time: Given A, we have

$\mathbb{F}^n = E_0 \oplus U$, where both are invariant, A is nilpotent on E_0, and A is injective ($\Rightarrow$ bijective) on U.

Prop 1) The only eigenvalue of A on E_0 is 0

2) 0 is not an eigenvalue of A on U.

Pf 1) If $A\bar{v} = \lambda\bar{v}$, then $A^m\bar{v} = \lambda^m\bar{v}$. Thus a non-zero eigenvector for $\lambda \neq 0$ is one on which A does not act nilpotently.

$\Rightarrow$ none in E_0.

2) 0 is an eigenvalue $\longleftrightarrow$ $\ker(A) \neq \{\bar{0}\}$. A is injective on U.

Now a small piece of notation. If U is an A-invariant subspace, then multiplication by A is a linear operator on U. We'll denote this operator by $A|_U$.

Cor If $p_A(\lambda) = \lambda^{m_0}(\lambda - \lambda_1)^{m_1}\cdots$, then

$p_{A|_{E_0}}(\lambda) = \lambda^{m_0}$ and $p_{A|_U}(\lambda) = (\lambda-\lambda_1)^{m_1}\cdots$

Thus $\dim E_0 = m_0$.

Pf By invariance, we have a block decomposition

$$A \sim \left[\begin{array}{c|c} A_0 & 0 \\ \hline 0 & A_1 \end{array}\right] \qquad A_0 = A|_{E_0}, \quad A_1 = A|_U.$$

So $p_A(\lambda) = \left|\begin{array}{c|c} A_0-\lambda I & 0 \\ \hline 0 & A_1-\lambda I \end{array}\right| = \det(A_0-\lambda I)\cdot\det(A_1-\lambda I)$

By the previous prop, only roots of $\det(A_0-\lambda I)$ are zero & zero not a root of $\det(A_1-\lambda I)$

So, up to a sign, $\det(A_0-\lambda I) = \lambda^m$

$\det(A_1-\lambda I) = p_A(\lambda)/\lambda^m$.

Since $\dim V = \deg p_A(\lambda)$, $\dim E_0 = m$. $\boxtimes$

Def Let λ_0 be an eigenvalue of A. The generalized eigenspace for λ_0, E_{λ_0}, is $\bigcup \ker((A-\lambda I)^m)$.

Lemma If $B = A-\lambda_0 I$, then $p_B(\lambda) = p_A(\lambda+\lambda_0)$.

Pf $p_B(\lambda) = \det(B-\lambda I) = \det(A-\lambda_0 I-\lambda I)$

$= \det(A-(\lambda+\lambda_0)I) = p_A(\lambda+\lambda_0)$ $\boxtimes$

Prop $\dim E_{\lambda_0}$ = multiplicity of λ_0 as a root in $p_A(\lambda)$.

Pf Apply the previous theorem to $A-\lambda_0 I$. Then if $B = A-\lambda_0 I$, then E_0 for B is E_{λ_0} for A. Thus $\dim E_{\lambda_0}$ = multiplicity of 0 as a root of $p_B(\lambda)$. This is the multiplicity of λ_0 as a root of $p_A(\lambda)$. $\boxtimes$

We are now in a position to generalize the splitting into eigenspaces.

Thm If $p_A(\lambda) = (\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_r)^{m_r}$, then $V = E_{\lambda_1}\oplus\cdots\oplus E_{\lambda_n}$ and $\dim E_{\lambda_k} = m_k$.

Pf By induction on r. The base case is obvious

By our previous propositions, we know that

$V = U \oplus E_{\lambda_r}$, and U is A-invariant & $(A-\lambda_r I)$ is an isomorphism on U. Then the char poly of A on U is $p_{A|_U}(\lambda) = (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_{r-1})^{m_{r-1}}$. Now on U, we can apply the inductive hypothesis $\Rightarrow$

$$U = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_{r-1}}$$

$$\Rightarrow V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}. \qquad \boxed{}$$

Remark In fact, the E_{λ_k} are the ones named above: $\dim(E_{\lambda_k}) = m_k$ & the E_{λ_k} in U is a subspace of $\mathbb{F}^n$ w/ same dimension and contained in E_{λ_k}. So it must be E_{λ_k}.

This theorem makes Jordan form work. The direct sum decomposition tells us that A is similar to a matrix:

$$\begin{array}{cc} \left[\begin{array}{c|c|c|c} A_{\lambda_1} & 0 & 0 & 0 \\ \hline 0 & A_{\lambda_2} & 0 & 0 \\ \hline 0 & 0 & \ddots & \vdots \\ \hline 0 & 0 & \cdots & A_{\lambda_r} \end{array}\right] & \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_r \end{array} \\ \begin{array}{cccc} m_1 & m_2 & \cdots & m_r \end{array} & \end{array}$$

& moreover, the behavior of A_{λ_k} is determined by the nilpotent matrix $(A_{\lambda_k} - \lambda_k \cdot I)$.

So if we can find a nice form for $B = (A_{\lambda_k} - \lambda_k \cdot I)$, say J, then the corresponding one for $A_{\lambda_k} = J + \lambda_k \cdot I$.

We look now at a nilpotent matrix A, acting on $\mathbb{F}^n$.

So we have a sequence of subspaces

$$\{\vec{0}\} \subseteq \ker(A) \subseteq \ker(A^2) \subseteq \cdots \subseteq \ker(A^m) = \mathbb{F}^n,$$

and we also know that for $0 < k \leq m$,

$$\ker(A^{k-1}) \subsetneq \ker(A^k)$$

Prop $A \cdot \ker(A^k) \subseteq \ker(A^{k-1})$

Pf If $\bar{v} \in \ker(A^k)$, then $\bar{0} = A^k \bar{v} = A^{k-1} \cdot A\bar{v}$

$$\Rightarrow A\bar{v} \in \ker(A^{k-1}) \qquad \boxtimes$$

Now we can get a sequence of linearly independent elements, each lying in distinct strata.

Prop If $\bar{v} \in \ker(A^k) - \ker(A^{k-1})$, then $\{\bar{v}, A\bar{v}, \ldots, A^{k-1}\bar{v}\}$ is linearly independent.

Pf By induction on k, and the base case is obvious. So assume that for all $\bar{w} \in \ker(A^{k-1}) - \ker(A^{k-2})$, $\{\bar{w}, \ldots, A^{k-2}\bar{w}\}$ is linearly independent, and consider a linear dependence relation

$$a_0 \bar{v} + \ldots + a_{k-1} A^{k-1} \bar{v} = \bar{0}$$

Applying A, and letting $\bar{w} = A\bar{v}$, we get

$$a_0 \bar{w} + \ldots + a_{k-2} A^{k-2} \bar{w} = \bar{0}.$$

By the induction hypothesis, $\{\bar{w}, \ldots, A^{k-2}\bar{w}\}$ is lin. ind, so $a_0 = \cdots = a_{k-2} = 0$. Thus our original equality was

$$a_{k-1} A^{k-1}\bar{v} = \bar{0}.$$

Since $\bar{v} \notin \ker(A^{k-1})$, we conclude $a_{k-1} = 0$ & so $\{\bar{v}, \ldots, A^{k-1}\bar{v}\}$ is linearly independent. $\boxtimes$

On the subspace $\langle \bar{v}, \ldots, A^{k-1}\bar{v} \rangle$, we have an easy matrix for A:

$$A \simeq \begin{array}{c} \\ \begin{array}{cccc} A^{k-1}\bar v & A^{k-2}\bar v & \cdots & \bar v \end{array} \\ \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A^{k-1}\bar v \\ A^{k-2}\bar v \\ \vdots \\ \bar v \end{array}$$

<u>Def</u> The $k\times k$ - <u>Jordan Block matrix</u> (w/ eigenvalue λ) is

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & \ddots \\ 0 & & & 1 \\ & & & \lambda \end{bmatrix}$$

So on a space like $\langle \bar v, A\bar v, \dots, A^{k-1}\bar v\rangle$, with the obvious basis, $A \sim J_k(0)$.

We can do better.

<u>Prop</u> Let $\bar v_1, \dots, \bar v_s \in \ker(A^k)$ be such that $\{A^{k-1}\bar v_1, \dots, A^{k-1}\bar v_s\}$ is linearly independent. Then $\{\bar v_1, \dots, \bar v_s, A\bar v_1, \dots, A\bar v_s, \dots, A^{k-1}\bar v_1, \dots, A^{k-1}\bar v_s\}$ is lin. ind.

<u>Pf</u> By induction on k, and the base case is again obvious. Consider the linear dependence relation

$$(*) \qquad \begin{aligned} &a_{11}\bar v_1 + \dots + a_{1s}\bar v_s + \\ &a_{21}A\bar v_1 + \dots \qquad + \qquad = \bar 0. \\ &\quad a_{ij}A^{i-1}\bar v_j + \dots \end{aligned}$$

If we apply A and let $\bar w_i = A\bar v_i$, then $(*)$ becomes

$$\begin{aligned} &a_{11}\bar w_1 + \dots + a_{1s}\bar w_s + \\ &\vdots \qquad\qquad\qquad\qquad = \bar 0. \\ &+ a_{k-1,1}A^{k-2}\bar w_1 + \dots + a_{k-1,s}A^{k-2}\bar w_s \end{aligned}$$

Our inductive hypothesis ensures that $\{\bar w_1, \dots, A^{k-2}\bar w_s\}$ is independent. Thus

$a_{ij} = 0$ for $i < k$. Now $(*)$ is

$$a_{k1} A^{k-1} \bar{v}_1 + \dots + a_{ks} A^{k-1} \bar{v}_s = \bar{0},$$

and we assumed $\{A^{k-1} \bar{v}_1, \dots, A^{k-1} \bar{v}_s\}$ was lin. ind. Thus $a_{ki} = 0$ for all i.

And hence $\{\bar{v}_1, \dots, A^{k-1} \bar{v}_s\}$ is lin. independent. $\boxtimes$