LECTURE 8 - DUALS CONTINUED

We saw last time that to any vector space V, we can naturally associate another vector space, the dual space V^* of linear functional $V \to \mathbb{F}$. This had the property that if $L\colon V \to W$, then there is a linear map $L^*\colon W^* \to V^*$ defined simply by composition, and this assignment takes subspaces to quotient spaces and quotient spaces to subspaces. We also saw how to find the matrix of the dual to a linear transformation.

Proposition. *If V and W are finite dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{C}$ respectively and if we let the duals have the dual bases, then*

$$_{\mathcal{B}^*}[L^*]_{\mathcal{C}^*} = \left({}_{\mathcal{C}}[L]_{\mathcal{B}}\right)^t.$$

The heart of this assignment is that the transpose takes row vectors to column vectors, and we should think of the dual space to the space of column vectors as being the space of row vectors. Then we are simply noting that

$$\vec{x} \cdot (A \cdot \bar{y}) = (\vec{x} \cdot A) \cdot \bar{y},$$

where $\vec{x}$ is a row vector and $\bar{y}$ is a column vector. If we translate the part in parentheses in the right-hand side into a column vector using the transpose, then we get

$$(\vec{x} \cdot A)^t = A^t \cdot \vec{x}^t,$$

which is just a restatement of the above.

We pause here to make a remark about our finite dimensional hypothesis. We saw that if we choose a basis for V, then we get a dual basis for V^*. This has the same number of elements, so we conclude that choosing a basis $\{\bar{v}_1, \dots\}$ for a finite dimensional V induces an isomorphism

$$V \to V^*, \bar{v}_i \mapsto \delta_{\bar{v}_i}.$$

This isomorphism is *unnatural*: it requires a choice of basis, rather than a nice intrinsic description. It does, however, show something very nice: for finite dimensional vector spaces, every subspace is dual to a quotient and every quotient is dual to a subspace. We'll see in a moment a related isomorphism that is natural, and when we talk about bilinear forms and inner product spaces, we'll see a way to make this isomorphism natural.

Let's continue to make things slightly more concrete (and follow-up with an example). We begin with an observation for finite dimensional V. Assume that f is a non-zero linear functional. This means that there is some $\bar{v} \in V$ such that $f(\bar{v}) \neq 0$. Thus f is a surjective linear transformation to a 1-dimensional vector space. The rank nullity theorem says then that the kernel is a $(\dim(V) - 1)$-dimensional subspace of V, and we have a natural short exact sequence

$$0 \to \ker(f) \to V \xrightarrow{f} \mathbb{F} \to 0.$$

This shows us that the essential feature of f is its kernel. We'll spell that out more clearly through two propositions.

Proposition. *If $\bar{x}$ is any vector such that $f(\bar{x}) \neq 0$, then*

$$V = \langle \bar{x} \rangle \oplus \ker(f).$$

Proof. We have to show that these two subspaces intersect trivially and span V. For the former, if $a\bar{x} \in \ker(f)$, then

$$0 = f(a\bar{x}) = af(\bar{x}).$$

Since $f(\bar{x}) \neq 0$, we can divide both sides by it and conclude that $a = 0$. Thus $a\bar{x} = \bar{0}$.

To show the two spaces span, we let $\bar{u} \in V$. Then $\big(f(\bar{u})/f(\bar{x})\big)\bar{x} \in \langle \bar{x} \rangle$, while a direct computation shows

$$\bar{u} - \big(f(\bar{u})/f(\bar{x})\big)\bar{x} \in \ker(f).$$

Since $\bar{u}$ is obviously the sum of these, we see that $V = \langle \bar{x} \rangle + \ker(f)$. □

This is a surprisingly useful little proposition. In particular, we can use it to show that up to scaling, the kernel of a functional determines the functional.

Proposition. *If f and g are not identically zero and* $\ker(f) = \ker(g)$*, then there is some constant λ such that $f(\bar{v}) = \lambda g(\bar{v})$ for all $\bar{v} \in V$.*

Proof. Use the previous proposition to write V as $\langle \bar{x} \rangle \oplus \ker(f)$, where $f(\bar{x}) \neq 0$. This means that for any $\bar{v} \in V$, we can write $\bar{v}$ uniquely as $a\bar{x} + \bar{k}$, where $f(\bar{k}) = 0$, and so

$$f(\bar{v}) = f(a\bar{x} + \bar{k}) = af(\bar{x}).$$

Since $\ker(g) = \ker(f)$, we see that

$$g(\bar{v}) = g(a\bar{x} + \bar{k}) = ag(\bar{x}),$$

and since g is assumed to not be identically zero, we know that $g(\bar{x}) \neq 0$. Now let $\lambda = f(\bar{x})/g(\bar{x})$. Then on $\ker(f)$, f and λg agree, and on $\langle \bar{x} \rangle$, f and λg agree. Thus f and λg agree everywhere! □

Thus not only do we see that there is a naturally associated $(\dim(V) - 1)$-dimensional subspace associated to f, we know that up to scalar multiples, it completely determines f.

Now an example.

Let $V = \mathbb{R}^3$, and let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be $f(a\bar{e}_1 + b\bar{e}_2 + c\bar{e}_3) = a + b + c$. Then we can easily compute that $\ker(f) = \langle \bar{e}_1 - \bar{e}_2, \bar{e}_2 - \bar{e}_3 \rangle$, and $\bar{e}_1$ is a vector such that $f(\bar{e}_1) \neq 0$. Thus we know

$$\mathbb{R}^3 = \langle \bar{e}_1 \rangle \oplus \langle \bar{e}_1 - \bar{e}_2, \bar{e}_2 - \bar{e}_3 \rangle.$$

However, we could have chosen any vector such that f evaluated to a non-zero value on it to produce such a direct sum decomposition. In particular, we could have chosen $\bar{e}_1 + \bar{e}_2 + \bar{e}_3$, or any linear combination such that $a + b + c \neq 0$.

Finally, we can express f in terms of the dual basis. Recall that in general

$$f = f(\bar{e}_1)\delta_{\bar{e}_1} + f(\bar{e}_2)\delta_{\bar{e}_2} + f(\bar{e}_3)\delta_{\bar{e}_3}.$$

Thus we need only evaluate f on the given basis vectors. Our f evaluates to 1 everywhere, so we get

$$f = \delta_{\bar{e}_1} + \delta_{\bar{e}_2} + \delta_{\bar{e}_3}.$$

Now for a second, less familiar example. Let f be defined the same way, but now as a map $(\mathbb{F}_3)^3 \to \mathbb{F}_3$. Then we can find other vectors in the kernel than we might expect: $\bar{e}_1 + \bar{e}_2 + \bar{e}_3 \in \ker(f)$. The vectors we had before are also still in the kernel, so we must have a linear dependence relation between them. In fact

$$(\bar{e}_1 - \bar{e}_2) + 2(\bar{e}_2 - \bar{e}_3) = \bar{e}_1 + \bar{e}_2 - 2\bar{e}_3,$$

and this was our desired vector. Notice now that we can't use this as a choice of vector on which f is non-zero. A homework problem is to determine how many of the 27 vectors in $\mathbb{F}_3^3$ can be used as $\bar{x}$ in our direct sum decomposition relative to f.

As was discusses, our isomorphism between V and its dual is not natural: we had to choose a basis. If we iterate the construction of the dual,then the story changes considerable.

Defintion. *Let* V^{**} *denote* $\big(V^*\big)^*$, *the double dual.*

Theorem. *If* V *is finite dimensional, then the natural map* $V \to V^{**}$ *given by*

$$\bar{v} \mapsto \big(f \mapsto f(\bar{v})\big)$$

is an isomorphism.

A word of explanation: the value on $\bar{v}$ of the linear map $V \to V^{**}$ is a linear map $V^* \to \mathbb{F}$. This then is a rule that assigns to each functional a number. We have a great one: evaluate the functional on $\bar{v}$, and that's all this map is.

Proof. These are both vector spaces of the same dimension, so it will suffice to show that the map is injective. Assume $\bar{f}$ is in the kernel. Then $f(\bar{v}) = 0$ for all $f \in V^{**}$. In particular, $\delta_{\bar{v}_i}(\bar{v}) = 0$ for all dual basis vectors $\delta_{\bar{v}_i}$. These however pull out the coefficient of $\bar{v}_i$ in the linear combination expressing $\bar{v}$, so we conclude that

$$\bar{v} = 0\bar{v}_1 + \cdots = \bar{0}.$$

□

Though this proof made use of a basis, it was only a rhetorical device. What makes this map special is that it is an isomorphism between V and V^{**} that makes no use of a particular basis. We can do better.

Proposition. *If* $L\colon V \to W$, *then the following diagram commutes*

$$\begin{array}{ccc} V & \longrightarrow & V^{**} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L^{**}} \\ W & \longrightarrow & W^{**} \end{array} ,$$

where the unlabeled maps are the natural isomorphisms in the previous theorem.

So then what happens when V is infinite dimensional? Lots of stuff just fails horribly.

Proposition. *If* $\dim(V) = \infty$, *then* $\dim(V^*) > \dim(V)$, *and they are never isomorphic. In particular,* V *is never isomorphic to its double dual in the infinite dimensional case.*

As stated, this is true. The problem is a lack of hypotheses to make sure that the size of the dual is bounded. When we talk about bilinear forms and Hilbert spaces, we can refine this to give a larger class of vector spaces for which the space is canonically isomorphic to its double dual.