Lecture 6 - Quotient Spaces

Last time: defined "equivalent" linear transformations.
Start with a refinement if \( L : V \rightarrow Y \), rather than \( L : V \rightarrow W \). Here we can make do with only one basis, since \( V = W \).

What happens if we change basis? \( B, B' \)

\[
[B]_B = P B B [L]_B B P_B',
\]

\[
= B P_B B [L]_B B (P_B')^{-1}.
\]

**Def:** \( A \) and \( B \) in \( M_n(\mathbb{F}) \) are **similar** if there is an invertible \( Q \in M_n(\mathbb{F}) \) s.t.

\[ A = Q B Q^{-1}. \]

This is an equivalence relation, denoted \( \sim \).

So similar matrices correspond to different forms of the same linear transformation.

It's much hicke to find good representatives for equivalence classes. This is the point of Jordan form.

The rank-nullity theorem is the start of a rather beautiful story: quotient spaces. First a little about \( X/\sim \)

**Prop:** If \( X, \sim \) is a set with an equiv. relation \( \sim \) \( f : X \rightarrow Y \) is a map such that

if \( x \sim y \), then \( f(x) = f(y) \), then we have a natural extension \( X/\sim \rightarrow Y \).
such that $\xrightarrow{f} \xrightarrow{\sim} \xrightarrow{\sim f} \xrightarrow{X/\sim} Y$ commutes.

This is an example of a universal property; it characterizes $\xrightarrow{X/\sim} Y$.

**Pf.** Define $\tilde{f}(\tilde{x})$ by $\tilde{f}(\tilde{x}) = f(x)$. If we chose a different representative, say $y$, then since $f(x) = f(y)$, $\tilde{f}$ has the same value and hence is well-defined.

We apply this concept to vector spaces.

**Def.** If $W \subseteq V$, we define an equivalence relation $\sim_W$ by $\tilde{v} \sim_W \tilde{w}$ iff $\tilde{v} - \tilde{w} \in W$.

**Prop.** This is an equivalence relation.

**Pf.** Since $W$ is a subspace, $0 = \tilde{v} - \tilde{w} \in W \forall \tilde{v}, \tilde{w}$.

Let $\tilde{v} \sim_W \tilde{w}$ and $\tilde{w} \sim_W \tilde{u}$, then $\tilde{v} - \tilde{w} \in W \Rightarrow -(\tilde{v} - \tilde{w}) = \tilde{w} - \tilde{v} \in W$ (so $\tilde{w} \sim_W \tilde{v}$)

and $\tilde{v} - \tilde{w}, \tilde{v} - \tilde{u} \in W \Rightarrow (\tilde{v} - \tilde{w}) + (\tilde{w} - \tilde{u}) = \tilde{v} - \tilde{u} \in W$, so $\tilde{v} \sim_W \tilde{u}$.

**Def.** The equivalence class of $\tilde{v}$ is the **coset** of $\tilde{v}$ and is denoted $\tilde{v} + W$.

**Remark.** $\tilde{v} \sim_W \tilde{w}$ means $\tilde{v} - \tilde{w} \in W$, so there is a $\tilde{u} \in W$ s.t. $\tilde{v} = \tilde{w} + \tilde{u}$. This means anything equivalent to $\tilde{v}$ is of the form $\tilde{v} + \tilde{u}$ for some $\tilde{u} \in W$, or $\|\tilde{v}\| = \|\tilde{u}\|$ as sets.

**Def.** The **quotient space** $V/\sim_W$ is the set of equivalence classes $\tilde{v} + W$.
Geometric Example: $\mathbb{R}^2 = V$, $(x,y)$-plane $= W$. Then the coset $\vec{v} + (xy)$ is the translate of the $(x,y)$-plane passing through the tip of $\vec{v}$.

Similarly, if $\vec{v} \in (x,y)$-plane, then we haven't done anything. Finally, one way to parametrize the cosets is via the z-coordinate of $\vec{v}$. That doesn't change.

Example suggests $V/W$ is a vector space. Here's how.

Prop If $\vec{v} \sim \vec{v}'$ and $\vec{u} \sim \vec{u}'$, then $a\vec{v} + b\vec{u} \sim a\vec{v}' + b\vec{u}'$.

Pf $\vec{v} \sim \vec{v}'$ means $\vec{v}' = \vec{v} + \vec{w}$, $\vec{w} \in W$, and similarly for $\vec{u}, \vec{u}'$. Thus $a\vec{v}' + b\vec{u}' = a(\vec{v} + \vec{w}) + b(\vec{u} + \vec{w}) = (a\vec{v} + b\vec{u}) + (a\vec{w} + b\vec{w})$, so $a\vec{v}' + b\vec{u}' \sim a\vec{v} + b\vec{u}$. $\square$

This has a powerful consequence.

Prop $a \cdot (\vec{v} + W) + b \cdot (\vec{u} + W) = (a \cdot \vec{v} + b \cdot \vec{u}) + W$ endows $V/W$ with a vector space structure $\&$

$\Pi_W: V \rightarrow V/W$ is linear.

$\overrightarrow{\vec{v}} \mapsto \vec{v} + W$.

The second part in some sense defines the first.

Pf That the operations are well-defined is a consequence of the previous proposition: linear combinations of equivalent things are equivalent. Verifying the axioms is routine.

That the map is linear is then obvious. $\square$

Def The map $\Pi_W: V \rightarrow V/W$ is the canonical projection.
Thm: If \( L: V \to U \) has \( W \leq \ker(L) \), then there is a natural extension \( \tilde{L}: V/W \to U \) making \( V \xrightarrow{L} U \) commute.

Proof: Again, \( \tilde{L}(\pi_W(v)) = L(v) \), so there is only one way to define it. Since \( W \leq \ker(L) \), \( \tilde{L} \) is well-defined:

\[
\tilde{L}(\pi_W(v)) = \tilde{L}(\pi_W(\bar{v})) = \tilde{L}(\bar{v}) = L(\bar{v}) = L(v).
\]

It's also clearly linear:

\[
\tilde{L}(a(\bar{v} + \bar{w}) + b(\bar{v} + \bar{w})) = \tilde{L}(a\bar{v} + b\bar{w}) = L(a\bar{v} + b\bar{w}) = aL(\bar{v}) + bL(\bar{w})
\]

\( \square \)

We'll use this implicitly to understand the isomorphism theorems that follow.

Remark: Up to isom, \( V/W \) is characterized by the previous result. Any other \( V \to U \) that satisfies the conditions must be, up to isomorphism, \( V \to V/W \).

Thm: \( \pi_W \) establishes a bijection between subspaces of \( V \) containing \( W \) and subspaces of \( V/W \).

Proof: If \( U \leq V/W \) is a subspace, then \( U = \{ \bar{v} + W \mid \bar{v} \in S \} \), and \( \pi_W U = \{ \bar{v} + W \mid \bar{v} \in S, \bar{u} \in W \} \) is a subspace containing \( W \). \( \pi_W \) of this is clearly \( U \). The other direction is similar.

Thus we get a kind of induction on dimension: if we can prove something for \( V/W \) and for \( W \), then we can boot-strap to all of \( V \). Now a big theorem.

Thm (1st Isomorphism Theorem): If \( L: V \to U \), then \( L \) induces an isomorphism \( \tilde{L}: V/\ker(L) \to \text{Im}(L) \).
Pf: Since \( \ker(L) = \ker(L) \), we have a natural map \( \tilde{L} \)

\[ \tilde{L} : \frac{V}{\ker(L)} \rightarrow U. \]

If \( \tilde{w} \in \im(L) \), then \( \tilde{w} = L(\tilde{v}) = \tilde{L}(\tilde{v} + 1z_{\ker(L)}) \), so

\[ \im(\tilde{L}) = \im(L). \]

Since every map factors as

\[ V \rightarrow \im(L) \subseteq U, \]

we conclude

\[ \tilde{L} : \frac{V}{\ker(L)} \rightarrow \im(L) \text{ is onto.} \]

If \( \tilde{w} = \tilde{L}(\tilde{v} + 1z_{\ker(L)}) = L(\tilde{v}) \), then \( \tilde{v} \in \ker(L) \) and \( \tilde{v} + 1z_{\ker(L)} = \ker(L) \),

the zero vector in \( \frac{V}{\ker(L)} \). Thus \( \tilde{L} \) is injective. \( \Box \)

Cor: \( \dim(V) = \dim(W) + \dim(\frac{V}{W}) \) for any subspace \( W \).

This is the theorem the rank-nullity theorem wants to be.

Everything today has been basis independent (this is why the words "natural" and "canonical" have come from). If we have a basis, then we can make other statements:

Choosing a basis for the complement of \( W \) gives us a section \( \frac{V}{W} \xrightarrow{S} V \) such that \( \pi_W \circ S = \text{Id}_{\frac{V}{W}} \). This gives us a non-canonical isomorphism \( W \oplus \frac{V}{W} \cong V \). We'll return to this next time.