Lecture 6 - Quotient Spaces

Last time: defined "equivalent" linear transformations.
Start with a refinement if $L: V \to V$, rather than $L: V \to W$. Here we can make do with only one basis, since $V = W$.
What happens if we change basis? $\mathcal{B}, \mathcal{B}'$

$$
\begin{aligned}
{}_{\mathcal{B}'}[L]_{\mathcal{B}'} &= {}_{\mathcal{B}'}P_{\mathcal{B}}\,{}_{\mathcal{B}}[L]_{\mathcal{B}}\,{}_{\mathcal{B}}P_{\mathcal{B}'} \\
&= {}_{\mathcal{B}'}P_{\mathcal{B}}\,{}_{\mathcal{B}}[L]_{\mathcal{B}}\left({}_{\mathcal{B}'}P_{\mathcal{B}}\right)^{-1}.
\end{aligned}
$$

<u>Def</u> A and B in $M_n(\mathbb{F})$ are <u>similar</u> if there is an invertible $Q \in M_n(\mathbb{F})$ s.t.

$$A = QBQ^{-1}.$$

This is an equivalence relation, denoted $\sim$.
So similar matrices correspond to different forms of the same linear transformation.
It's much trickier to find good representatives for equivalence classes. This is the point of Jordan form.

The rank-nullity theorem is the start of a rather beautiful story: quotient spaces. First a little about $X/\sim$

<u>Prop</u> If $X, \sim$ is a set with an equiv. relation & $f: X \to Y$ is a map such that if $x \sim y$, then $f(x) = f(y)$, then we have a natural extension $X/\sim \;\to Y$

such that
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ X/\sim \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$
commutes.

This is an example of a universal property; it characterizes $X \xrightarrow{\pi} X/\sim$.

Pf: define $\tilde{f}([x])$ by $\tilde{f}([x]) = f(x)$. If we chose a different representative, say y, then since $f(x) = f(y)$, $\tilde{f}$ has the same value and hence is well-defined. $\boxtimes$

We apply this concept to vector spaces.

Def If $W \subseteq V$, we define an equivalence relation $\sim_W$ by $\bar{v} \sim_W \bar{w}$ iff $\bar{v} - \bar{w} \in W$.

Prop This is an equivalence relation.

Pf. Since W is a subspace, $0 = \bar{v} - \bar{v} \in W$ $\forall \bar{v}$, & if $\bar{v} \sim_W \bar{w}$ and $\bar{w} \sim_W \bar{u}$, then

$\bar{v} - \bar{w} \in W \Rightarrow -(\bar{v} - \bar{w}) = \bar{w} - \bar{v} \in W$ (so $\bar{w} \sim_W \bar{v}$)

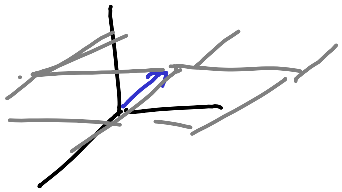
and $\bar{v} - \bar{w}, \bar{w} - \bar{u} \in W \Rightarrow (\bar{v} - \bar{w}) + (\bar{w} - \bar{u}) = \bar{v} - \bar{u} \in W$, so $\bar{v} \sim_W \bar{u}$. $\boxtimes$

Def The equivalence class of $\bar{v}$ is the coset of V and is denoted $\bar{v} + W$.

Remark $\bar{v} \sim_W \bar{w}$ means $\bar{v} - \bar{w} \in W$, so there is a $\bar{u} \in W$ s.t. $\bar{v} = \bar{w} + \bar{u}$. This means anything equivalent to $\bar{v}$ is of the form $\bar{v} + \bar{u}$ for some $\bar{u} \in W$, or $[\bar{v}] = \bar{v} + W$ as sets.

Def The quotient space V/W is the set of equivalence classes $V/\sim_W$.

Geometric Example: $\mathbb{R}^3 = V$, (x,y)-plane $= W$. Then the coset $\bar{v} + (x,y)$ is the translate of the (x,y)-plane passing through the tip of $\bar{v}$.



Similarly, if $\bar{v} \in (x,y)$-plane, then we haven't done anything. Finally, one way to parametrize the cosets is via the z-coord of $\bar{v}$. That doesn't change.

Example suggests V/W is a vector space. Here's how

Prop If $\bar{v} \sim \bar{v}'$ and $\bar{u} \sim \bar{u}'$, then $a\bar{v} + b\bar{u} \sim a\bar{v}' + b\bar{u}'$.

Pf $\bar{v} \sim \bar{v}'$ means $\bar{v}' = \bar{v} + \bar{w}_1$, $\bar{w}_1 \in W$, and similarly for $\bar{u}, \bar{u}'$. Thus

$$a\bar{v}' + b\bar{u}' = a(\bar{v} + \bar{w}_1) + b\cdot(\bar{u} + \bar{w}_2) = (a\bar{v} + b\bar{u}) + (\underbrace{a\bar{w}_1 + b\bar{w}_2}_{\in W}).$$

So $a\bar{v}' + b\bar{u}' \sim a\bar{v} + b\bar{u}$. $\boxtimes$

This has a powerful consequence.

Prop $a\cdot(\bar{v} + W) + b\cdot(\bar{u} + W) = (a\cdot\bar{v} + b\cdot\bar{u}) + W$ endows V/W with a vector space structure & $\pi_W : V \longrightarrow V/W$, $\bar{v} \longmapsto \bar{v} + W$, is linear.

The second part in some sense defines the first.

Pf That the operations are well-defined is a consequence of the previous proposition: linear combinations of equivalent things are equivalent. Verifying the axioms is routine. That the map is linear is then obvious. $\boxtimes$

Def The map $\pi_W : V \longrightarrow V/W$ is the canonical projection.

Thm If $L: V \to U$ has $W \subseteq \ker(L)$, then there is a natural extension $\tilde{L}: V/W \to U$ making

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi_W"'] & U \\ V/W \arrow[ur, "\tilde{L}"'] & \end{tikzcd}$$

commute.

Pf Again, $\tilde{L}(\pi_W(\bar{v})) = L(\bar{v})$, so there is only one way to define it. Since $W \subseteq \ker(L)$, $\tilde{L}$ is well-defined:

$\bar{x} \sim_W \bar{y}$, then $\bar{x} = \bar{y} + \bar{w}$, so

$L(\bar{x}) = L(\bar{y}+\bar{w}) = L(\bar{y}) + \cancel{L(\bar{w})}^{0} = L(\bar{y})$. It's also clearly linear:

$\tilde{L}(a\cdot(\bar{v}+W) + b\cdot(\bar{u}+W)) = \tilde{L}((a\bar{v}+b\bar{u})+W) = L(a\bar{v}+b\bar{u}) = aL(\bar{v}) + bL(\bar{u})$

$= a\tilde{L}(\bar{v}+W) + b\tilde{L}(\bar{u}+W)$. ▨

We'll use this implicitly to understand the isomorphism theorems that follow.

Remark Up to isom, V/W is characterized by the previous result. Any other $V \to T$ that satisfies the conditions must be, up to isomorphism, $V \to V/W$.

Thm: π_W establishes a bijection between subspaces of V containing W and subspaces of V/W.

Pf: If $U \subseteq V/W$ is a subspace, then $U = \{\bar{s}+W \mid s \in S\}$, and $\pi_W^{-1} U = \{\bar{s}+\bar{w} \mid s \in S, \bar{w} \in W\}$ is a subspace containing W. π_W of this is clearly U. The other direction is similar.

Thus we get a kind of induction on dimension: if we can prove something for V/W and for W, then we can boot-strap to all of V. Now a big theorem

Thm (1[st] Isomorphism Theorem) If $L: V \to U$, then L induces an isomorphism $\tilde{L}: V/\ker(L) \xrightarrow{\cong} \operatorname{Im}(L)$.

Pf: Since $\ker(L)=\ker(L)$, we have a natural map $\tilde{L}$

$$\tilde{L}: V/\ker(L) \longrightarrow U.$$

If $\bar{w} \in \operatorname{Im}(L)$, then $\bar{w} = L(\bar{v}) = \tilde{L}(\bar{v} + \ker(L))$, so $\operatorname{Im}(\tilde{L}) = \operatorname{Im}(L)$. Since every map factors as

$$V \longrightarrow \operatorname{Im}(L) \subseteq U,$$ we conclude

$$\tilde{L}: V/\ker(L) \longrightarrow \operatorname{Im}(L) \quad \text{is onto.}$$

If $\bar{0} = \tilde{L}(\bar{v} + \ker(L)) = L(\bar{v})$, then $\bar{v} \in \ker(L)$ and $\bar{v} + \ker(L) = \ker(L)$, the zero vector in $V/\ker(L)$. Thus $\tilde{L}$ is injective. ☒

Cor: $\dim(V) = \dim(W) + \dim(V/W)$ for any subspace W.

This is the theorem the rank-Nullity theorem wants to be.

Everything today has been basis independent (this is where the words "natural" and "canonical" have come from). If we have a basis, then we can make other statements:

Choosing a basis for the complement of W gives us a section $V/W \xrightarrow{S} V$ such that $\pi_W \circ S = Id_{V/W}$. This gives us a non-canonical isomorphism $W \oplus V/W \cong V$. We'll return to this next time.