Lecture 5 - More on Linear Maps

Last time we saw the rank-nullity theorem:

$$\dim U = \underbrace{\dim(\ker L)}_{\text{nullity}} + \underbrace{\dim(\operatorname{Im} L)}_{\text{rank}}$$

The terms come from matrices. If A is a matrix, then the null-space of A is the set of solutions to the homogeneous system

$$A \cdot \bar{x} = \bar{0}.$$

The rank is the dimension of the space of those $\bar{b}$ such that $A \cdot \bar{x} = \bar{b}$ has a solution.

This has another name: the column space.

Our analysis shows that with a choice of basis, the column space of ${}_{C}[L]_{B}$ corresponds to the image of L & the null-space corresponds to the kernel. So these really do reproduce our earlier concepts.

Now let's see that our map ${}_{C}[-]_{B}$ is an isomorphism

It's mono: If $L \in \ker(\;)$, then $L(\bar{b}_i) = \bar{0}$ for all i (since this pulls out the coefficients. Thus $L = 0$.

It's epi: Consider $\phi_C \circ A \circ \phi_B^{-1} : U \to V$. This is a composite of linear maps. Then ${}_{C}[\phi_C \circ A \circ \phi_B^{-1}]_{B} = A$.

We evaluate on $\bar{b}_i$ & apply ϕ_C^{-1} to get a column vector

$$\phi_C^{-1} \circ (\phi_C \circ A \circ \phi_B^{-1})(\bar{b}_i) = A(\bar{e}_i) = i^{th} \text{ column of } A. \qquad \boxtimes$$

Thus a choice of basis everywhere identifies U with $\mathbb{F}^n$ and $L(U,V)$ with $m \times n$-matrices.

Can ask how a choice of basis changes the form of this isomorphism.

<u>Prop</u> If $L: U \to V$ is an isomorphism, then the image under L of a basis of U is a basis of V.

<u>Remark</u>: Can do better: L a monomorphism $\Rightarrow$ (lin ind in U) $\to$ (lin ind in V) & L an epimorphism $\Rightarrow$ (spanning in U) $\rightsquigarrow$ (spanning in V).

<u>Pf</u>: Let $a_1 L(\bar{v}_1) + \dots a_n L(\bar{v}_n) = \bar{0}$. Then by linearity,

$$L(a_1\bar{v}_1 + \dots + a_n\bar{v}_n) = \bar{0}, \quad \text{so} \quad a_1\bar{v}_1 + \dots + a_n\bar{v}_n \in \ker(L) = \{\bar{0}\}.$$

Since the $\bar{v}_i$ are linearly ind, we conclude that

$$a_1 = \dots = a_n = 0.$$

If $\bar{w} \in V$, then since L is onto, $\bar{w} = L(\bar{v})$. Since $\{\bar{v}_i\}$ spans, $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$ & $\bar{w} = a_1 L(\bar{v}_1) + \dots + a_n L(\bar{v}_n)$. $\blacksquare$

In fact monomorphism $\longleftrightarrow$ "takes linearly independent sets to linearly independent sets".

On the other hand, given two bases of V, we have an automorphism of V: $\bar{b}_i \longleftrightarrow \bar{c}_i$

So we see a very close connection between pairs of bases and automorphisms:

basis + aut = new basis & basis + basis = aut.

Let's apply to a test case of matrices to see how to relate different bases.

Prop An $n \times n$-matrix is invertible iff nullity $= 0$ or iff rank $= n$.

This is a restatement of the above. We've just used that n linearly independent vectors in an n-dimensional space form a basis.

So $(n \times n)$ invertible matrices $\longleftrightarrow$ ordered bases of $\mathbb{F}^n$.

But how does this work? Columns of matrix are our new basis elements. Problem: still expressed in the original basis!

So we are taking our new basis vectors $(\bar{b}_1, \ldots, \bar{b}_n)$ and writing them as a linear combination of the standard basis vectors.

Let's make this slightly fancier. We have

$$\begin{array}{ccc} \mathbb{F}^n & \longrightarrow & \mathbb{F}^n \\ \bar{b}\text{-basis} & & \bar{e}\text{-basis} \\ \bar{b}_i & \longmapsto & \bar{b}_i \end{array}$$

& we are taking the associated matrix

This is a very special linear transformation: the identity! (Take $\bar{b}$ to $\ldots$ $\bar{b}$).

Thus A is the matrix

$$\begin{array}{cc} & \begin{array}{ccc} \bar{b}_1 & \cdots & \bar{b}_n \end{array} \\ \begin{array}{c} \bar{e}_1 \\ \vdots \\ \bar{e}_n \end{array} & \left[\begin{array}{ccc} & & \\ & I & \\ & & \end{array} \right] \end{array} = {}_{\mathcal{E}}[I]_{\mathcal{B}}$$

We'll call this a "change of basis" matrix and write it ${}_{\mathcal{E}}P_{\mathcal{B}}$.

Observation: For any $\mathcal{B}, \mathcal{C}, \mathcal{D}$,

$${}_{\mathcal{D}}P_{\mathcal{C}}\ {}_{\mathcal{C}}P_{\mathcal{B}} = {}_{\mathcal{D}}P_{\mathcal{B}}.$$

Thus ${}_{C}P_{B} = {}_{B}P_{C}^{-1}$ and

$${}_{\mathcal{C}}P_{B} = ({}_{\mathcal{E}}P_{\mathcal{C}})^{-1}\,{}_{\mathcal{E}}P_{\mathcal{B}}.$$

The last part is the most useful. It is easy to compute ${}_{\mathcal{E}}P_{B}$ (it's normally given to us in a description of $\mathcal{B}$), then we compute the inverses and multiply.

Why is this sort of thing true?

${}_{\mathcal{C}}P_{B}$ has columns which express $\bar{b}$ in the $\bar{c}$ basis.

${}_{\mathcal{D}}P_{\mathcal{C}}$ does similar things. So we start in the $\mathcal{B}$-basis, write things in the $\mathcal{C}$-basis, the substitute for $\bar{c}_i$ the expression in the $\mathcal{D}$-basis. We still just have $\bar{b}$, but now in the $\mathcal{D}$-basis.

This holds more generally:

Thm If $\mathcal{B}, \mathcal{B}'$ are ordered bases of U and $\mathcal{C}, \mathcal{C}'$ ordered bases of V, then

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = {}_{\mathcal{C}}P_{\mathcal{C}'} \cdot {}_{\mathcal{C}'}[L]_{\mathcal{B}'} \cdot {}_{\mathcal{B}'}P_{\mathcal{B}}$$

We just look at what both give. The ith column of ${}_e[L]_{\mathcal{B}}$ is the coefficient vector of $L(\bar{b}_i)$ expressed in the e-basis.

Similarly, ${}_{\mathcal{B}'}P_{\mathcal{B}}$ expresses $\bar{b}_i$ as a linear combination of the $\bar{b}'$s.

$$\bar{b}_i = a_1\bar{b}_1' + \dots + a_n\bar{b}_n'$$

Then we multiply by ${}_{e'}[L]_{\mathcal{B}'}$, which amounts to evaluating L on the vector and writing the result in the e' basis. This is <u>still</u> just $L(\bar{b}_i)$, but in the e' basis. Multiplying by ${}_eP_{e'}$ amounts to substituting in the e-basis ie. expressing $L(\bar{b}_i)$ in the e basis.

Now an aside.

<u>Def</u> An <u>equivalence</u> <u>relation</u> is a relation $\sim$ on X s.t.

1) $a \sim a$ (reflexive)
2) $a\sim b \Rightarrow b \sim a$ (symmetric)
3) $a\sim b, b\sim c \Rightarrow a \sim c$ (transitive)

These are "like" equality but in general a little weaker.
(Recall: a relation R is a subset of $X\times X$ & aRb means $(a,b)\in R$. So it's a truth statement).

<u>Def</u> The <u>equivalence</u> <u>class</u> of $x \in X$ is

$$[x] = \{y \mid x \sim y\}.$$

The quotient $X/\sim$ is the set of equivalence classes.

In general, equivalence classes are disjoint or equal:

Prop $[x] \cap [y] = \begin{cases} [x] & x \sim y \\ \emptyset & x \not\sim y \end{cases}$

This is obviously symmetric.

Assume $x \sim y$. Then $y \sim x$ by symmetry & $x \in [y]$. Now if $z \in [x]$, then $x \sim z$, so $y \sim z$ & $z \in [y]$. Thus $[x] \subseteq [y]$. Swapping roles gives $[y] \subseteq [x]$. Conversely, if $z \in [x] \cap [y]$, then $x \sim z$ & $y \sim z$, so $x \sim y$ & in first case. $\boxtimes$

Ex: On vector spaces, "is isomorphic to" is an equiv relation.

$$\left\{ \begin{matrix} \text{finite dimensional} \\ \text{v. spaces} \end{matrix} \right\} \xrightarrow{\dim} \mathbb{N}$$ is a bijection,

since every f.d. vector space is isomorphic to $\mathbb{F}^n$ for some n & dim is the isomorphism invariant.

Ex: On $M_{m \times n}(\mathbb{F})$, say A and B are equivalent if we can find a $P \in M_{m \times m}(\mathbb{F})$ and $Q \in M_{n \times n}(\mathbb{F})$, both invertible, such that

$$A = PBQ.$$

Prop This is an equivalence relation.

1) $A = I \cdot A \cdot I$, so $A \sim A$
2) $A = PBQ \Rightarrow P^{-1} A Q^{-1} = B$ so $A \sim B \Rightarrow B \sim A$
3) $A = PBQ$ & $B = RCT$, then $A = (PR) C (TQ)$ & $(PR)^{-1} = R^{-1} P^{-1}$, etc, so $A \sim B, B \sim C \Rightarrow A \sim C$. $\boxtimes$

If we identify P and Q with change-of-basis matrices, then we see this equivalence relation identifies two matrices if they have the same underlying linear transformation.

We can find easy representatives of each equiv. class. Let $L: U \to V$, and as in the proof of the rank-nullity theorem, write $U = U' \oplus \ker(L)$, where $L|_{U'}$ is an isomorphism onto the image. Choose bases $\{\bar{u}_1, \dots, \bar{u}_k\}$ of U' and $\{\bar{u}_{k+1}, \dots, \bar{u}_n\}$ of $\ker(L)$, and since $L|_{U'}$ is injective, $\{L\bar{u}_1, \dots, L\bar{u}_k\}$ is linearly independent. Extend this to a basis $\mathcal{C}$ of V. Now

$$
{}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{array}{c} \\ L\bar{u}_1 \\ \vdots \\ L\bar{u}_k \\ \bar{v}_{k+1} \\ \vdots \\ \bar{v}_m \end{array}
\begin{array}{c} \begin{array}{cccccc} \bar{u}_1 & \cdots & \bar{u}_k, & \bar{u}_{k+1} & \cdots & \bar{u}_n \end{array} \\
\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}.
$$

From this we can easily read out everything.

<u>Remark</u> We can stay entirely in matrices and get this. Elementary row operations = left mult by invertible things allows us to proceed to reduced row eschelon form. Elementary column ops = right mult $\Rightarrow$ the form above.