# LECTURE 3

(TYPED!)

**Defintion.** *A <u>basis</u> is a linearly independent spanning set.*

**Theorem.** *Every vector space has a basis.*

We won't prove this; it's actually essentially equivalent in the infinite dimensional case to one of the axioms of set theory: the axiom of choice. Instead, we'll accept this as given.

Things are slightly easier if we assume that $V$ has a finite basis.

**Theorem.** *If $V$ has a finite basis, then if $I$ is a linearly independent set and $S$ is a spanning one, then $|I| \leq |S|$. In particular, all linearly independent sets in $V$ are finite.*

*Proof.* Since $V$ has a finite basis, when we show that $|I|$ is at most the cardinality of that basis, then we will have shown that $I$ is a finite set. We may therefore assume that everything in sight is finite. Enumerate the elements of $I$ as $\bar{v}_1, \dots, \bar{v}_n$ and those of $S$ as $\bar{w}_1, \dots, \bar{w}_m$. Since $S$ spans, we can express $\bar{v}_1$ as a linear combination of elements of $S$, and since $\bar{v}_1 \neq \bar{0}$, we know some $\bar{w}_i$ has a non-zero coefficient. By reordering, we may assume it's $\bar{w}_1$. However, this means that $\langle \bar{v}_1, \bar{w}_2, \dots \rangle = \langle S \rangle$, since we could just solve for $\bar{w}_1$ in the linear combination expressing $\bar{v}_1$ as an element of the span. Iterating this, we express $\bar{v}_k$ as a linear combination of $\bar{v}_1, \dots, \bar{v}_{k-1}, \bar{w}_k, \dots$, and since $I$ is independent, we know that for some $j$, the coefficient of $\bar{w}_j$ is non-zero. Again, we may assume that $j = k$, and we can replace $\bar{w}_k$ with $\bar{v}_k$. If there are few $\bar{w}$s than $\bar{v}$s, then we eventually have a spanning set $\bar{v}_1, \dots, \bar{v}_m$. However, $\bar{v}_{m+1}$ is not an element of the span, since $I$ was independent, and this provides a contradiction. $\square$

**Corollary.** *Every basis has the same cardinality.*

*Proof.* Each is a linearly independent, spanning set, and we can just have them alternate roles for the previous theorem. $\square$

We remark that this theorem is also equally true in the infinite case.

**Defintion.** *The <u>dimension</u> of a vector space is the cardinality of any basis.*

This is a complete invariant of a vector space. To specify the dimension is to specify the isomorphism type! To make this precise, we need another notion of direct sum.

**Defintion.** *Let $I$ be a set, and let $V_i$ be a collection of vector spaces indexed by $I$. Then the <u>direct product</u> of the $V_i$ is defined to be*

$$\prod_{i \in I} V_i = \{f \colon I \to \bigcup_{i \in I} V_i | f(i) \in V_i\}.$$

Put into words, these are the functions from $I$ where the value at $i$ is a vector in $V_i$. Coordinatewise operations make this set into a vector space in a natural way.

**Defintion.** *Let $f$ and $g$ be elements of $\prod_{i\in I} V_i$. Then the linear combination $a \cdot f + b \cdot g$ is the function defined by*

$$(a \cdot f + b \cdot g)(i) = a \cdot f(i) + b \cdot g(i).$$

Since all of the $V_i$ are vector spaces, these operations make $\prod_{i\in I} V_i$ into a vector space. In general, this vector space is HUGE. The elements of $I$ don't assemble in any meaningful way to give a basis, and in fact, the direct product in general has a dimension bigger than the cardinality of $I$. The problem is with our requirement that we only have finite sums.

**Defintion.** *The support of an element in the direct product is*

$$Supp(f) = \{i \in I | f(i) \neq 0\}.$$

Those elements where the support is finite are those that we can get as linear combinations of vectors related to $I$. This forms an important subspace.

**Defintion.** *The direct sum of the collection $V_i$ is*

$$\bigoplus_{i\in I} V_i = \{f \in \prod_{i\in I} V_i || Supp(f)| < \infty\}.$$

**Proposition.** *The direct sum is a subspace of the direct product.*

This is much nicer. We have much better control on the size of the direct sum! We can use this to get another notion (although as stated, it won't be as canonical as we might like.

**Defintion.** *If $X$ is a set, then we define the free vector space on $X$, $\mathbb{F}X$ to be the direct sum over $X$ of copies of $\mathbb{F}$:*

$$\bigoplus_{x\in X} \mathbb{F}.$$

This vector space has a very nice basis.

**Proposition.** *Let $f_x$ denote the function*

$$f_x(y) = \begin{cases} 1 & y = x \\ 0 & otherwise. \end{cases}$$

*Then $\{f_x, x \in X\}$ forms a basis for $\mathbb{F}X$.*

*Proof.* This is almost obvious. These functions are clearly linearly independent (since each function involves only one "coordinate"). We need to see that they span. However, if we are given a function $f$ of finite support, then

$$f = \sum_{y\in Supp(f)} f(y) f_y.$$

This shows that the span of these functions is all of $\mathbb{F}X$. □

We will often denote $f_y$ by simply $y$, and instead of thinking of functions, think of "linear combinations of elements of $X$". Free vector spaces are closely tied to linear combinations (and this is somewhat why this definition is slightly unnatural). As we talk more about the structure of linear transformations, this will be made more clear. For now, we need the universal property.

**Theorem.** *Giving a linear transformation $L\colon \mathbb{F}X \to V$ is the same a giving a map of sets $X \to V$. Given a function $f\colon X \to V$, we associate the linear transformation*

$$L_f(a_1x_1 + \cdots + a_nx_n) = a_1f(x_1) + \cdots + a_nf(x_n).$$

*Given a linear transformation $L\colon \mathbb{F}X \to V$, we get a function on $X$ by restriction.*

What's this theorem saying? It says that if you have a free vector space, then you can describe any linear transformation by specifying the values on this distinguished basis. We of course know more is true: given any vector space and any basis, we can describe any linear transformation by specifying the values on a basis. We'll see that as a consequence of this.

*Proof.* By construction, we see that restricting $L_f$ to $X$ gives back the function $f$. That gives one direction. For the other, to specify a linear transformation $L\colon \mathbb{F}X \to V$, we just have to specify the values on a basis. This is by linearity: every element is a linear combination of basis elements, and linear transformations play nicely with linear combinations. This gives the other direction. □

Now we can get to an important restatement of an earlier theorem.

**Theorem.** *If $V$ is a vector space, then a choice of basis $\mathcal{B}$ gives us an isomorphism*

$$\mathbb{F}\mathcal{B} \to V.$$

*Thus every vector space is free.*

*Proof.* We saw that to give a function from $\mathcal{B}$ to $V$ is to give a linear transformation. We also have an obvious function: take each element of the basis to itself. We need to show that it is $1-1$ and onto.

"Onto" is a restatement that the basis spans. If we have a vector $\bar{v} \in V$, then we need to show it is a linear combination of elements of $\mathcal{B}$. Since $\mathcal{B}$ is a basis, we know this to be true:

$$\bar{v} = a_1\bar{v}_1 + \cdots + a_n\bar{v}_n,$$

and then we learn that

$$\bar{v} = L(a_1\bar{v}_1 + \cdots + a_n\bar{v}_n).$$

(We remark that our choice of notation for the basis and the indexing set makes this statement essentially obvious). Thus every element is in the image.

"1-1" is a restatement of linear independence. If

$$L(a_1\bar{v}_1 + \cdots + a_n\bar{v}_n) = L(b_1\bar{w}_1 + \cdots + b_m\bar{v}_m),$$

then in $V$,

$$a_1\bar{v}_1 + \cdots + a_n\bar{v}_n = b_1\bar{w}_1 + \cdots + b_m\bar{v}_m.$$

Since $\mathcal{B}$ was assumed to be linearly independent, we know that $n = m$, that up to reordering $\bar{v}_i = \bar{w}_i$, and that the corresponding coefficients are equal. That means the starting elements were equal. □

As stated, this seems like an empty theorem. The important fact is that the right-hand side is an arbitrary vector space. We know basically nothing about it. The left-hand side is a very simple vector space. If our basis is finite, it's just $\mathbb{F}^n$, where $n$ is the dimension. So what the theorem is really telling us is that all vector spaces are very simple.