LECTURE 2

Defintion. *A subset W of a vector space V is a subspace if*

1. *W is non-empty*
2. *For every $\bar{v}, \bar{w} \in W$ and $a, b \in \mathbb{F}$, $a\bar{v} + b\bar{w} \in W$.*

Expressions like $a\bar{v} + b\bar{w}$, or more generally

$$\sum_{i=1}^{k} a_i \bar{v} + i$$

are called linear combinations. So a non-empty subset of V is a subspace if it is closed under linear combinations. Much of today's class will focus on properties of subsets and subspaces detected by various conditions on linear combinations.

Theorem. *If W is a subspace of V, then W is a vector space over $\mathbb{F}$ with operations coming from those of V.*

In particular, since all of those axioms are satisfied for V, then they are for W. We only have to check closure!

Examples:

Defintion. *Let $\mathbb{F}^n = \{(a_1, \ldots, a_n) | a_i \in \mathbb{F}\}$ with coordinate-wise addition and scalar multiplication.*

This gives us a few examples. Let $W \subset \mathbb{F}^n$ be those points which are zero except in the first coordinate:

$$W = \{(a, 0, \ldots, 0)\} \subset \mathbb{F}^n.$$

Then W is a subspace, since

$$a \cdot (\alpha, 0, \ldots, 0) + b \cdot (\beta, 0, \ldots, 0) = (a\alpha + b\beta, 0, \ldots, 0) \in W.$$

If $\mathbb{F} = \mathbb{R}$, then $W' = \{(a_1, \ldots, a_n) | a_i \geq 0\}$ is not a subspace. It's closed under addition, but not scalar multiplication.

We have a number of ways to build new subspaces from old.

Proposition. *If W_i for $i \in I$ is a collection of subspaces of V, then*

$$W = \bigcap_{i \in I} W_i = \{\bar{w} \in V | \bar{w} \in W_i \forall i \in I\}$$

is a subspace.

Proof. Let $\bar{v}, \bar{w} \in W$. Then for all $i \in I$, $\bar{v}, \bar{w} \in W_i$, by definition. Since each W_i is a subspace, we then learn that for all $a, b \in \mathbb{F}$,

$$a\bar{v} + b\bar{w} \in W_i,$$

and hence $a\bar{v} + b\bar{w} \in W$. □

Thought question: Why is this never empty?

The union is a little trickier.

Proposition. *$W_1 \cup W_2$ is a subspace iff $W_1 \subset W_2$ or $W_2 \subset W_1$.*

Proof. $\Leftarrow$ is obvious. We need to show the other, so assume that we can find $\bar{w}_1 \in W_1 - W_2$ and $\bar{w}_2 \in W_2 - W_1$. Then if $W_1 \cup W_2$ is a subspace, then $\bar{w}_1 + \bar{w}_2 \in W_1 \cup W_2$. This means it's in one of them, and without loss of generality, we may assume it's in W_1. But this means

$$w_2 = (\bar{w}_1 + \bar{w}_2) - \bar{w}_1 \in W_1,$$

a contradiction. $\square$

Example to keep in mind: $\mathbb{R}^2$ with $W_1 = x$-axis and $W_2 = y$-axis.

Instead of the union, we consider the smallest subspace containing the union.

Defintion. *If W_1 and W_2 are subspaces of V, then the sum is*

$$W_1 + W_2 = \{a\bar{w}_1 + b\bar{w}_2 | \bar{w}_i \in W_i\}.$$

In other words, we consider all linear combinations of elements of W_1 and W_2. Clearly if W is any subspace that contains W_1 and W_2, then W contains $W_1 + W_2$. On the other hand, it's also closed under linear combinations and non-empty, so this is a subspace.

Special attention is given to the case where $W_1 \cap W_2 = \{\bar{0}\}$.

Defintion. *If $W_1 \cap W_2 = \{\bar{0}\}$, then we say the sum of W_1 and W_2 is the (internal) direct sum, and we write it $W_1 \oplus W_2$.*

So what's so special about direct sums?

Proposition. *Every element in $W_1 \oplus W_2$ can be uniquely written as $\bar{w}_1 + \bar{w}_2$.*

Proof. Assume $\bar{w}_1 + \bar{w}_2 = \bar{v}_1 + \bar{v}_2$, where the subscript indicates the subspace from which the element is drawn. Then rearranging, we see

$$\bar{w}_1 - \bar{v}_1 = \bar{v}_2 - \bar{w}_2.$$

The left-hand side is in W_1, while the right-hand side is in W_2, so both are in the intersection. This means both are $\bar{0}$, and hence $\bar{w}_i = \bar{v}_i$. $\square$

Our notion of sums therefore includes two distinct notions:

(1) being able to write elements as linear combinations
(2) being able to do so uniquely.

The former is related to span, the latter to linear independence.

Defintion. *Let $X \subset V$. Then the span of X is the collection of all linear combinations of elements of X:*

$$Span(X) = \langle X \rangle = \{a_1\bar{x}_1 + \cdots + a_n\bar{x}_n | \bar{x}_i \in X, n \geq 0\}.$$

If X is empty, then $\langle X \rangle = \{\bar{0}\}$.

The following is immediate.

Proposition. *The set $\langle X \rangle$ is a subspace of V.*

Examples: For any V, $\langle V \rangle = V$. If $X = W \cup U$, then $\langle X \rangle = W + U$.

Just as before, if W is a subspace of V and W contains X, then $\langle X \rangle \subset W$. Thus $\langle X \rangle$ is the smallest subspace containing X, and the elements of X provide convenient names for every element of their span.

Proposition. *If $\bar{w} \in \langle X \rangle$, then*

$$\langle \{\bar{w}\} \cup X \rangle = \langle X \rangle.$$

Proof. One inclusion is obvious. For the other, since $\bar{w} \in \langle X \rangle$, we know we can write it as a linear combination of elements of X. Thus if we have a linear combination of elements of X and $\bar{w}$, then we can substitute for $\bar{w}$ and get a linear combination with just elements of X. □

This helps us greatly in cutting out redundant elements: anything we can name using a smaller set of elements is extraneous.

Defintion. *A spanning set is a set X such that $\langle X \rangle = V$.*

In general, these can be very big. One of the goals of much of linear algebra is to give a very compact spanning set for an arbitrary vector space.

The corresponding small notion is linear independence.

Defintion. *A set X is linearly independent if*

$$a_1\bar{v}_1 + \cdots + a_n\bar{v}_n = \bar{0}$$

implies $a_1 = \cdots = a_n = 0$ for any $\bar{v}_i \in X$.

If X is not linearly independent, then it is linearly dependent.

We again see only $\bar{0}$ showing up. We can restate this definition as "$\bar{0}$ has a unique presentation as a linear combination of elements of X.

Theorem. *If X is linearly independent and*

$$a_1\bar{v}_1 + \ldots a_n\bar{v}_n = b_1\bar{w}_1 + \cdots + b_m\bar{w}_m,$$

where all vectors are from X and all coefficients are non-zero, then $n = m$ and up to reordering, $a_i = b_i$ and $\bar{v}_i = \bar{w}_i$ for all i.

Proof. Order the $\bar{v}$s and $\bar{w}$s so that $\bar{v}_1 = \bar{w}_1, \ldots, \bar{v}_t = \bar{w}_t$. Moving all terms to one side then gives

$$(a_1 - b_1)\bar{v}_1 + \cdots + (a_t - b_t)\bar{v}_t + a_{t+1}\bar{v}_{t+1} + \cdots - (b_{t+1}\bar{w}_{t+1} + \ldots) = \bar{0}.$$

Since X is assumed to be linearly independent, we learn that $a_{t+1} = \cdots = a_n = 0$ and $b_{t+1} = \cdots = 0$, and by our assumption on the non-zeroness of the coefficients, this forces $t = n = m$. Moreover, we learn that $a_i = b_i$ for all i. □

Thus knowing that $\bar{0}$ has a unique presentation as a linear combination ensures that everything in the span does so. We can use span more directly to test for linear independence.

Proposition. *A set X is linearly independent if and only if for all $\bar{v} \in X$, $\bar{v} \notin \langle X - \{\bar{v}\} \rangle$.*

Proof. We show the negative of this. X is linearly dependent if there is a linear dependence relation

$$a_1\bar{v}_1 + \cdots + a_n\bar{v}_n = \bar{0},$$

with some coefficient (say a_1) not equal to zero. Solving for $\bar{v}_1$ then expresses $\bar{v}_1$ as an element of the span of the remaining vectors. □

Thus if a set is linearly independent, then we can add any vector not in the span of it to it and still have a linearly independent set. On the other hand, if we add in a vector in the span, then the set becomes linearly dependent.