LECTURE 2

Definition. A subset $W$ of a vector space $V$ is a subspace if

1. $W$ is non-empty
2. For every $\bar{v}, \bar{w} \in W$ and $a, b \in \mathbb{F}$, $a\bar{v} + b\bar{w} \in W$.

Expressions like $a\bar{v} + b\bar{w}$, or more generally

$$\sum_{i=1}^{k} a_i \bar{v} + i$$

are called linear combinations. So a non-empty subset of $V$ is a subspace if it is closed under linear combinations. Much of today’s class will focus on properties of subsets and subspaces detected by various conditions on linear combinations.

Theorem. If $W$ is a subspace of $V$, then $W$ is a vector space over $\mathbb{F}$ with operations coming from those of $V$.

In particular, since all of those axioms are satisfied for $V$, then they are for $W$. We only have to check closure!

Examples:

Definition. Let $\mathbb{F}^n = \{(a_1, \ldots, a_n) | a_i \in \mathbb{F}\}$ with coordinate-wise addition and scalar multiplication.

This gives us a few examples. Let $W \subset \mathbb{F}^n$ be those points which are zero except in the first coordinate:

$$W = \{(a, 0, \ldots, 0) \} \subset \mathbb{F}^n.$$  

Then $W$ is a subspace, since

$$a \cdot (\alpha, 0, \ldots, 0) + b \cdot (\beta, 0, \ldots, 0) = (a\alpha + b\beta, 0, \ldots, 0) \in W.$$  

If $\mathbb{F} = \mathbb{R}$, then $W' = \{(a_1, \ldots, a_n) | a_i \geq 0\}$ is not a subspace. It’s closed under addition, but not scalar multiplication.

We have a number of ways to build new subspaces from old.

Proposition. If $W_i$ for $i \in I$ is a collection of subspaces of $V$, then

$$W = \bigcap_{i \in I} W_i = \{\bar{w} \in V | \bar{w} \in W_i \forall i \in I\}$$  

is a subspace.

Proof. Let $\bar{v}, \bar{w} \in W$. Then for all $i \in I$, $\bar{v}, \bar{w} \in W_i$, by definition. Since each $W_i$ is a subspace, we then learn that for all $a, b \in \mathbb{F}$,

$$a\bar{v} + b\bar{w} \in W_i,$$

and hence $a\bar{v} + b\bar{w} \in W$. \[
\]

Thought question: Why is this never empty? 
The union is a little trickier.

Proposition. $W_1 \cup W_2$ is a subspace iff $W_1 \subset W_2$ or $W_2 \subset W_1$. 

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**Proof.** ⇐ is obvious. We need to show the other, so assume that we can find \( \bar{w}_1 \in W_1 - W_2 \) and \( \bar{w}_2 \in W_2 - W_1 \). Then if \( W_1 \cup W_2 \) is a subspace, then \( \bar{w}_1 + \bar{w}_2 \in W_1 \cup W_2 \).

This means it’s in one of them, and without loss of generality, we may assume it’s in \( W_1 \). But this means

\[
\bar{w}_2 = (\bar{w}_1 + \bar{w}_2) - \bar{w}_1 \in W_1,
\]

a contradiction. \( \square \)

Example to keep in mind: \( \mathbb{R}^2 \) with \( W_1 = x\)-axis and \( W_2 = y\)-axis.

Instead of the union, we consider the smallest subspace containing the union.

**Definition.** If \( W_1 \) and \( W_2 \) are subspaces of \( V \), then the sum is

\[
W_1 + W_2 = \{a \bar{w}_1 + b \bar{w}_2 | \bar{w}_i \in W_i\}.
\]

In other words, we consider all linear combinations of elements of \( W_1 \) and \( W_2 \).

Clearly if \( W \) is any subspace that contains \( W_1 \) and \( W_2 \), then \( W \) contains \( W_1 + W_2 \). On the other hand, it’s also closed under linear combinations and non-empty, so this is a subspace.

Special attention is given to the case where \( W_1 \cap W_2 = \{ \bar{0} \} \).

**Definition.** If \( W_1 \cap W_2 = \{ \bar{0} \} \), then we say the sum of \( W_1 \) and \( W_2 \) is the (internal) direct sum, and we write it \( W_1 \oplus W_2 \).

So what’s so special about direct sums?

**Proposition.** Every element in \( W_1 \oplus W_2 \) can be uniquely written as \( \bar{w}_1 + \bar{w}_2 \).

**Proof.** Assume \( \bar{w}_1 + \bar{w}_2 = \bar{v}_1 + \bar{v}_2 \), where the subscript indicates the subspace from which the element is drawn. Then rearranging, we see

\[
\bar{w}_1 - \bar{v}_1 = \bar{v}_2 - \bar{w}_2.
\]

The left-hand side is in \( W_1 \), while the right-hand side is in \( W_2 \), so both are in the intersection. This means both are \( \bar{0} \), and hence \( \bar{w}_1 = \bar{v}_1 \). \( \square \)

Our notion of sums therefore includes two distinct notions:

1. being able to write elements as linear combinations
2. being able to do so uniquely.

The former is related to span, the latter to linear independence.

**Definition.** Let \( X \subset V \). Then the span of \( X \) is the collection of all linear combinations of elements of \( X \):

\[
\text{Span}(X) = \langle X \rangle = \{a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n | \bar{x}_i \in X, n \geq 0\}.
\]

If \( X \) is empty, then \( \langle X \rangle = \{ \bar{0} \} \).

The following is immediate.

**Proposition.** The set \( \langle X \rangle \) is a subspace of \( V \).

Examples: For any \( V \), \( \langle V \rangle = V \). If \( X = W \cup U \), then \( \langle X \rangle = W + U \).

Just as before, if \( W \) is a subspace of \( V \) and \( W \) contains \( X \), then \( \langle X \rangle \subset W \). Thus \( \langle X \rangle \) is the smallest subspace containing \( X \), and the elements of \( X \) provide convenient names for every element of their span.

**Proposition.** If \( \bar{w} \in \langle X \rangle \), then

\[
(\{ \bar{w} \} \cup X) = \langle X \rangle.
\]
Proof. One inclusion is obvious. For the other, since \( \bar{w} \in \langle X \rangle \), we know we can write it as a linear combination of elements of \( X \). Thus if we have a linear combination of elements of \( X \) and \( \bar{w} \), then we can substitute for \( \bar{w} \) and get a linear combination with just elements of \( X \). \( \square \)

This helps us greatly in cutting out redundant elements: anything we can name using a smaller set of elements is extraneous.

Definition. A spanning set is a set \( X \) such that \( \langle X \rangle = V \).

In general, these can be very big. One of the goals of much of linear algebra is to give a very compact spanning set for an arbitrary vector space.

The corresponding small notion is linear independence.

Definition. A set \( X \) is linearly independent if

\[
a_1 \bar{v}_1 + \cdots + a_n \bar{v}_n = \bar{0}
\]
implies \( a_1 = \cdots = a_n = 0 \) for any \( \bar{v}_i \in X \).

If \( X \) is not linearly independent, then it is linearly dependent.

We again see only \( \bar{0} \) showing up. We can restate this definition as “\( \bar{0} \) has a unique presentation as a linear combination of elements of \( X \).”

Theorem. If \( X \) is linearly independent and

\[
a_1 \bar{v}_1 + \cdots + a_n \bar{v}_n = b_1 \bar{w}_1 + \cdots + b_m \bar{w}_m,
\]
where all vectors are from \( X \) and all coefficients are non-zero, then \( n = m \) and up to reordering, \( a_i = b_i \) and \( \bar{v}_i = \bar{w}_i \) for all \( i \).

Proof. Order the \( \bar{v} \)s and \( \bar{w} \)s so that \( \bar{v}_1 = \bar{w}_1, \ldots, \bar{v}_t = \bar{w}_t \). Moving all terms to one side then gives

\[
(a_1 - b_1)\bar{v}_1 + \cdots + (a_t - b_t)\bar{v}_t + a_{t+1} \bar{v}_{t+1} + \cdots - (b_{t+1} \bar{w}_{t+1} + \cdots) = \bar{0}.
\]
Since \( X \) is assumed to be linearly independent, we learn that \( a_{t+1} = \cdots = a_n = 0 \) and \( b_{t+1} = \cdots = 0 \), and by our assumption on the non-zeroness of the coefficients, this forces \( t = n = m \). Moreover, we learn that \( a_i = b_i \) for all \( i \). \( \square \)

Thus knowing that \( \bar{0} \) has a unique presentation as a linear combination ensures that everything in the span does so. We can use span more directly to test for linear independence.

Proposition. A set \( X \) is linearly independent if and only if for all \( \bar{v} \in X \), \( \bar{v} \notin \langle X - \{\bar{v}\} \rangle \).

Proof. We show the negative of this. \( X \) is linearly dependent if there is a linear dependence relation

\[
a_1 \bar{v}_1 + \cdots + a_n \bar{v}_n = \bar{0},
\]
with some coefficient (say \( a_1 \)) not equal to zero. Solving for \( \bar{v}_1 \) then expresses \( \bar{v}_1 \) as an element of the span of the remaining vectors. \( \square \)

Thus if a set is linearly independent, then we can add any vector not in the span of it to it and still have a linearly independent set. On the other hand, if we add in a vector in the span, then the set becomes linearly dependent.