Lecture 1 - Introduction & Basic Definitions

I. Some Basic Notions

1. A **set** is a collection of objects.
2. The objects themselves are **elements**: \( x \in A \)
3. We then have lots of set operations:
   - **Union** \( \cup \): \( A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \)
   - **Intersection** \( \cap \): \( A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \)
   - **Product** \( \times \): \( A \times B = \{ (a, b) \mid a \in A, b \in B \} \)
   - **Difference** \( - \): \( A - B = \{ x \mid x \in A \text{ and } x \notin B \} \)
   - **Subset** \( \subseteq \): \( A \subseteq X \iff x \in A \Rightarrow x \in X \)
   - **Complement** \( ^c \): \( A^c = \{ x \in X \mid x \notin A \} \)
   - **Cardinality** \( | \): \( |A| = \text{number of elements in } A \)

4. **Functions** are rules assigning values in one set to elements of another \( f: A \rightarrow B \)
   - So subspaces \( \Gamma \) of \( A \times B \) s.t. for all \( a \in A \), there is a unique \( b \in B \) s.t. \( (a, b) \in \Gamma \)

We'll need a few more notions later.

II. Main Players

Linear algebra is the study of vector spaces. A first course works over the real or complex numbers. We'll look at a general field.

**Moral** Everything works the same as over the reals!
Defn A **field** is a set \( \mathbb{F} \) together with two operations:

\[ + : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \quad \& \quad \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \]

satisfying

\begin{align*}
\text{a1) } (a+b)+c &= a+(b+c) \\
\text{a2) } \exists 0 \in \mathbb{F} \text{ s.t. } a+0 = a = 0+a \forall a \\
\text{a3) } \forall a, \exists (-a) \text{ s.t. } a+(-a) = (-a)+a = 0 \\
\text{a4) } a+b = b+a \\
\text{a5) } (a+b) \cdot c = (a \cdot c) + (b \cdot c), \quad a \cdot (b+c) = (a \cdot b) + (a \cdot c) \\
\text{m1) } a \cdot (b \cdot c) = (a \cdot b) \cdot c \\
\text{m2) } \exists 1 \text{ s.t. } a \cdot 1 = 1 \cdot a = a \\
\text{m3) } \forall a \neq 0, \exists a^{-1} \text{ s.t. } a \cdot a^{-1} = a^{-1} \cdot a = 1 \\
\text{m4) } a \cdot b = b \cdot a.
\end{align*}

So \( (\mathbb{F}, +) \) is an abelian group \( \frac{1}{2} \), \( (\mathbb{F}, \cdot, \cdot) \) is an abelian group too.

**Basic Examples**

1) \( \mathbb{R} \) = real numbers
2) \( \mathbb{C} \) = complex numbers
3) \( \mathbb{Q} \) = rational numbers

**More Exotic Examples**

1) \( \mathbb{Q} \) = \( \{ x \in \mathbb{C} \mid x \text{ is a root of a poly w/ coefs in } \mathbb{Q} \} \)
2) Let \( \mathbb{F}_p = \{ 0, \ldots, p-1 \} \) \( \frac{1}{2} \) let

\[ a+b = \text{ remainder when } a \cdot b \text{ is divided by } p \text{ (and sim for \( \cdot )} \]

Then \( \mathbb{F}_p \) is a field.

**Ex**

\( \mathbb{F}_2: \{ 0, 1 \} \quad 1+1 = 0 \)
\( \mathbb{F}_3: \{ 0, 1, 2 \} \quad 1+2 = 0, \quad 2+2 = 1 = 2 \cdot 2 \)
Def A field $\mathbb{F}$ has **characteristic** $r$ if

$$1 + \ldots + 1 = 0 \text{, if no such } r \text{ exists, } \mathbb{F} \text{ has char } 0.$$ 

**Ex**: $\mathbb{F}_p$ has characteristic $p$  
$\mathbb{Q}$ has characteristic 0

This doesn't affect the vector spaces, but it does the fields.

Def A **vector space** over a field $\mathbb{F}$ is a set $V$ w/ 2 operations

$$+: V \times V \rightarrow V \quad \text{(vector addition)}$$ $$\cdot: \mathbb{F} \times V \rightarrow V \quad \text{(scalar multiplication)}$$

Satisfying:

a) $(V, +)$ is an abelian group  

m1) $(a \cdot b) \cdot \vec{v} = a \cdot (b \cdot \vec{v})$  

m2) $1 \cdot \vec{v} = \vec{v}$

d1) $(a+b) \cdot \vec{v} = a \cdot \vec{v} + b \cdot \vec{v}$  

d2) $a \cdot (\vec{v} + \vec{w}) = a \cdot \vec{v} + a \cdot \vec{w}$

So it has an addition $+$ a mult, $\cdot$ they are compatible.

We'll always denote vectors w/ a bar: $\vec{v}$

Saw in a previous course that vector spaces are essentially simple. Some is true here.

Def Let $V$ and $W$ be vector spaces. A **linear transformation** is a function $L: V \rightarrow W$ s.t.

$$L(a\vec{v} + b\vec{w}) = aL(\vec{v}) + bL(\vec{w}).$$

Linear transformations respect the structure.
Depending on the properties of the function, it can have other names:

- $f$ is 1-1: monomorphism
- $f$ is onto: epimorphism
- $f$ is bijective: isomorphism

**Thm** Every vector space is isomorphic to $\mathbb{F}^n$ for some $n$. This course will go beyond this simple statement.

We’ll focus more on the structure of v.s. without falling back on a particular basis.

**III. Outline**

A. General Review of Linear Algebra
B. Quotients ꞌ Isomorphism Theorems
C. Jordan Canonical Form ꞌ Friends
D. Bilinear ꞌ multilinear forms: Tensor ꞌ exterior products
E. Duals ꞌ Representability
F. Hilbert, Banach, ꞌ Topological Vector spaces