Lecture 1 – Introduction & Basic Definitions

I. Some Basic Notions

- A set is a collection of objects.
- The objects themselves are elements : $x \in A$

We then have lots of set operations:

Union $\cup$: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection $\cap$: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Product $\times$: $A \times B = \{(a,b) \mid a \in A,\ b \in B\}$

Difference $-$: $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Subset $\subseteq$: $A \subseteq X \longleftrightarrow x \in A \Rightarrow x \in X$

Complement ^{c} : $A \subseteq X$, $A^c = \{x \in X \mid x \notin A\}$

Cardinality $|\,|$: $|A|$ = number of elements in A

- Functions are rules assigning values in one set to elements of another $f: A \to B$

  (so subspaces Γ of $A \times B$ s.t. for all $a \in A$, there is a unique $b \in B$ s.t. $(a,b) \in \Gamma$)

We'll need a few more notions later.

II. Main Players

Linear algebra is the study of vector spaces. A first course work over the real or complex numbers. We'll look at a general field.

Moral Everything works the same as over the reals!

<u>Defn</u> A <u>field</u> is a set $\mathbb{F}$ together with two operations:

$+ : \mathbb{F}\times\mathbb{F} \to \mathbb{F}$ & $\cdot : \mathbb{F}\times\mathbb{F} \to \mathbb{F}$, satisfying

a1) $(a+b)+c = a+(b+c)$

a2) $\exists\, 0 \in \mathbb{F}$ s.t. $0+a = a+0 = a \;\forall a$

a3) $\forall a, \exists (-a)$ s.t $a+(-a) = (-a)+a = 0$

a4) $a+b = b+a$

d1) $(a+b)\cdot c = (a\cdot c)+(b\cdot c)$, $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$

m1) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

m2) $\exists 1$ s.t $a\cdot 1 = 1\cdot a = a$

m3) $\forall a \neq 0, \exists a^{-1}$ s.t. $a\cdot a^{-1} = a^{-1}\cdot a = 1$

m4) $a\cdot b = b\cdot a$.

Group

abelian group

unital ring

division ring

So $(\mathbb{F},+)$ is an abelian group & $(\mathbb{F}-\{0\},\cdot)$ is an abelian group too.

<u>Basic Examples</u>

1) $\mathbb{R}$ = real numbers
2) $\mathbb{C}$ = complex numbers
3) $\mathbb{Q}$ = rational numbers

<u>More Exotic Examples</u>

1) $\overline{\mathbb{Q}} = \{x \in \mathbb{C} \mid x \text{ is a root of a poly w/ coefs in } \mathbb{Q}\}$

2) Let $\mathbb{F}_p = \{0, \dots, p-1\}$ & let

$a+b$ = remainder when $a+b$ is divided by p (and sim for $\cdot$)

Then $\mathbb{F}_p$ is a field.

<u>Ex</u> $\mathbb{F}_2 : \{0,1\}$ $\quad 1+1=0$

$\mathbb{F}_3 : \{0,1,2\}$, $\quad 1+2=0$, $\quad 2+2=1=2\cdot 2$

Def A field $\mathbb{F}$ has characteristic r if

$\underbrace{1+\dots+1}_{r} = 0$. If no such r exists, $\mathbb{F}$ has char 0.

Ex: $\mathbb{F}_p$ has characteristic p

$\mathbb{Q}$ has characteristic 0

This doesn't affect the vector spaces, but it does the fields.

Def A vector space over a field $\mathbb{F}$ is a set V w/ 2 operations

$+ : V \times V \to V$ (vector addition)

$\cdot : \mathbb{F} \times V \to V$ (scalar multiplication)

satisfying:

a) $(V, +)$ is an abelian group

m1) $(a \cdot b) \cdot \bar{v} = a \cdot (b \cdot \bar{v})$

m2) $1 \cdot \bar{v} = \bar{v}$

d1) $(a+b) \cdot \bar{v} = a \cdot \bar{v} + b \cdot \bar{v}$

d2) $a \cdot (\bar{v} + \bar{w}) = a \cdot \bar{v} + a \cdot \bar{w}$

So it has an addition & a mult, & they are compatible

We'll always denote vectors w/ a bar: $\bar{v}$

Saw in a previous course that vector spaces are essentially simple. Some is true here.

Def Let V and W be vector spaces. A linear transformation is a function $L: V \to W$ s.t.

$$L(a\bar{v} + b\bar{w}) = aL(\bar{v}) + bL(\bar{w}).$$

Linear transformations respect the structure.

Depending on the properties of the function, L can have other names:

f is 1-1 : monomorphism

f is onto : epimorphism

f is bijective : isomorphism

Thm Every vector space is isomorphic to $\mathbb{F}^n$ some n.

This course will go beyond this simple statement. We'll focus more on the structure of v.s. without falling back on a particular basis.

III. Outline

A. General Review of Linear Algebra

B. Quotients & Isomorphism Theorems

C. Jordan Canonical Form & Friends

D. Bilinear & multilinear forms: Tensor & exterior products

E. Duals & Representability

F. Hilbert, Banach, & Topological vector spaces