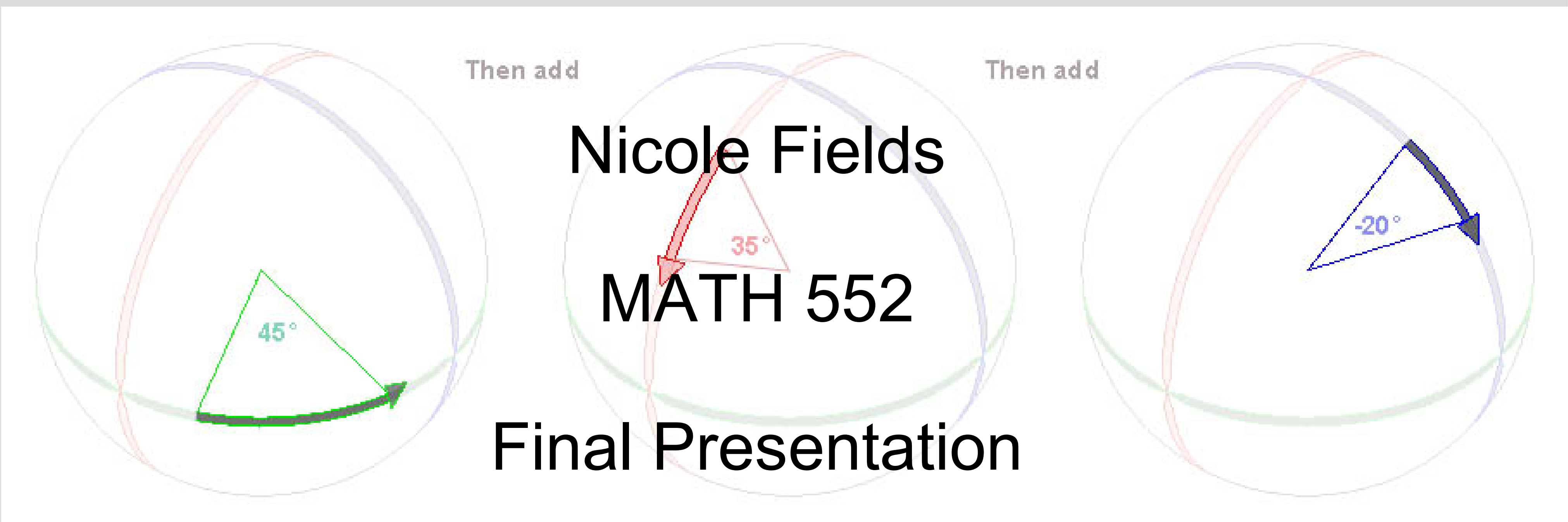
SU(2), SO(3) and SU(3)



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Group Definition

- SU(2) is the group of all 2 x 2 unitary matrices with determinant 1, elements are Complex
- SU(3) is the group of all 3 x 3 unitary matrices with determinant 1, elements are Complex
- Unitary matrix U, $U^{\dagger}U=I$
- SO(3) is the group of all 3 x 3 orthogonal matrices with determinant 1, elements are Real
- Orthogonal matrix O, $O^{T}O=I$
- The multiplication is basic matrix multiplication
- SU(2), SO(3), and SU(3) are all Lie groups, so they are both groups and manifolds

Group Structure

- SU(2), SO(3), and SU(3) are all topologically compact and simply connected
- They are simple Lie groups, which means that the only normal proper subgroup that they contain is the trivial one
- Being a simple Lie group also means that the associated Lie algebra is simple and that the Lie group is non-Abelian
- That SO(3) is non-Abelian leads to the non-commutative nature of rotations

Lie Algebras and Dimension

- Every Lie Group has an associated Lie Algebra
- They are related via the exponential map
- exp(Lie Algebra)=Lie Group
- The Lie Algebra can be represented by a number of infinitesimal generators or matrices that generate the Lie Group
- The number of matrices needed to generate a Lie Group is the same as the dimension of the group
- SU(n) groups have dimension n^2-1
- SO(n) groups have dimension n(n-1)/2
- SU(2), SO(3) dim=3; SU(3) dim=8

SU(2) and the Pauli Matrices

- The infinitesimal generators of SU(2) are $i\sigma_i$ where σ_1, σ_2, and σ_3 are the Pauli matrices
- The Pauli matrices are Hermitian, that is, they have real eigenvalues, and traceless

$$\sigma_1 = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \sigma_2 = \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \qquad \sigma_3 = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$$

- For a spin 1/2 particle, like an electron or a proton, the spin operators are $S_i = \hbar/2 * \sigma_i$

Pauli Matrices Commutation

- The Pauli matrices do not commute
- Physically this is because one cannot simultaneously measure spin in more than one direction
- Mathematically this is because SU(2) is non-Abelian
- The commutation relation is $\sigma_i\sigma_j=2i\varepsilon_{ijk}\sigma_k$

SU(2) and SO(3)

- The Lie Algebras of SU(2) and SO(3) are isomorphic
- This means that SU(2) and SO(3) are LOCALLY isomorphic
- This DOES NOT mean that SU(2) and SO(3) are isomorphic
- SU(2) is actually a double cover of SO(3) and there is a $2 \rightarrow 1$ surjective homeomorphism from SU(2) to SO(3)
- Spin 1/2 particles, or fermions, need to be rotated 720º in order to come back to the same state

SU(3) and the Gell-Mann Matrices

- By analogy to SU(2), SU(3) has infinitesimal generators $i\lambda_i$ where λ_i are the Gell-Mann matrices
- The Gell-Mann matrices are a generalization of the Pauli matrices, and they have similar properties
- There are eight 3 x 3 matrices which can be a representation of the 8 gluons that mediate Quantum Chromodynamics (QCD), also known as the strong force
- These matrices act on vectors with three elements that represent the three color charges associated with the strong force

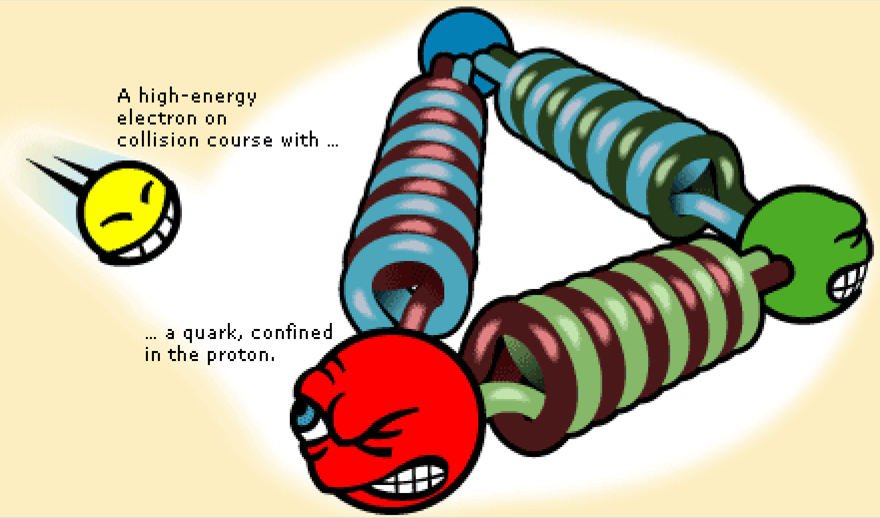
Gell-Mann Matrices

$$\lambda_1 = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \lambda_2 = \begin{matrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \lambda_3 = \begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\lambda_4 = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad \lambda_5 = \begin{matrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{matrix} \qquad \lambda_6 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

$$\lambda_7 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{matrix} \qquad \lambda_8 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix} * 1/\mathrm{sqrt}(3)$$

SU(2), SO(3), and SU(3)



- These three groups play a large role in physics, particle physics in particular
- SO(3) is used to describe and calculate external rotations
- SU(2) and SU(3) are used to describe and calculate internal rotations, while SU(2) deals with systems with two states, and SU(3) deals with systems with three states