Lecture 9 – Symmetric Groups

Note Title 2/14/2008

Prop 34 If $|G| = p^2 \cdot q$, then G has a normal s.g & is the semi-direct prod of H a p-Sylow & K a q-sylow. (p, q prime)

$K \cong \mathbb{Z}/q$

Pf $p > q$: $[G:H] = q$ is the smallest prime dividing $|G|$. Cor 24 $\Rightarrow$ $H \triangleleft G$.

$\Rightarrow HK$ is a s.g & $H \triangleleft$ $\Rightarrow$ G is $H \rtimes K$.

($H \cap K = \{e\}$: $|H \cap K|$ divides $\underset{p^2}{|H|}$ & $\underset{q}{|K|}$ $\Rightarrow$ $|H \cap K| = 1$)

$q > p$: K is a q-Sylow.

\# of q-Sylow: $n = 1 + k \cdot q$ & divides $p^2 \cdot q = |G|$.

$n \equiv 1 \ (q)$ & $n \mid p^2 \cdot q$ $\Rightarrow$ $n \mid p^2$

$n = 1$ ← K normal.

$n = p$ ← $p < q$, $n = 1 + kq \neq p$

$n = p^2$ ← $1 + kq = p^2$ $\Rightarrow$ $k \cdot q = (p+1)(p-1)$.

If q divides $(p+1)(p-1)$ $\Rightarrow$ $q = p+1$

$\Rightarrow p = 2, q = 3$.

Worry about having 4 3-Sylow s.g.

K_1, K_2, K_3, K_4.

$$K_i \cap K_j = \begin{cases} \{e\} & i \neq j \\ K_i = K_j & i = j \end{cases} \qquad |K_i \cap K_j| = \begin{cases} 1 \\ 3 \end{cases}$$

$$|K_1 \cup K_2 \cup K_3 \cup K_4| = 9$$

$\Rightarrow$ All elements of order dividing 4 (ie everything in H or its conj) are in the remaining 3 elements + $\{e\}$.

$\Rightarrow H \triangleleft G$. $|gHg^{-1}| = |H| = 4$ ▨

$k \in gHg^{-1}$, $k \to o(k) \mid 4$

Symmetric Groups:

- Cycles
- sign
- conjugacy classes

Def An element $\sigma \in S_n$ is a <u>cycle</u> if there are elements $i_1, \dots, i_r \in \{1, \dots, n\}$ such that $\sigma(i_1) = i_2, \dots, \sigma(i_{r-1}) = i_r, \sigma(i_r) = i_1$ and $\sigma(j) = j$ for $j \notin \{i_1, \dots, i_r\}$. In this case, r is the <u>length</u> and we write $\sigma = (i_1 \dots i_r)$.

$\leftrightarrow$ A cycle is a function determined entirely by its value on one element.

$$: \quad (i_1\ i_2 \cdots i_r) \longleftrightarrow f(j) = \begin{cases} j & j \notin \{i_1, \dots, i_r\} \\ i_{k+1} & j = i_k \\ i_1 & j = i_r \end{cases} \qquad 1 \le k \le r-1$$

$(1\ 3\ 8)$:

$1 \mapsto 3$ $\quad 2 \mapsto 2$

$3 \mapsto 8$ $\quad 4 \mapsto 4$

$8 \mapsto 1$ $\quad 5 \mapsto 5$

$\vdots$

Prop 35: If $\{i_1, \dots, i_s\}$ & $\{j_1, \dots, j_r\}$ are disjoint then $a = (i_1 \cdots i_s)$ & $b = (j_1 \cdots j_r)$ commute.

a & b are disjoint.

Pf: $ab(j)$ $\qquad ba(j)$

If $j \notin \{i_1, \dots, i_s, \dots j_r\}$, then $b(j) = j$ & $a(j) = j \Rightarrow ab(j) = j = ba(j)$

$i \in$ " , then either $a(i) = i$ or $b(i) = i$

not both.

$a(i) = i \Rightarrow i \in \{j_1, \dots, j_r\}$

Disjointness $\Rightarrow a(b^k(i)) = b^k(i)$

$b(i) \in \{j_1, \dots, j_r\} \Rightarrow b^k(i) \in \{ \dots \}$

$\Rightarrow b(i) \notin \{i_1, \dots, i_s\} \Rightarrow a$ fixes $b(i)$.

$a(b(i)) = b(i) = b(a(i))$. $\blacksquare$

Thm 36 Any element of S_n can be written uniquely as a product of disjoint cycles.

Pf: Let i_1 be the 1st integer moved by $c \in S_n$.

$(i_1\ c(i_1) \cdots c^{k_1}(i_1))$ is a cycle some k_1 s.t. $c^{k_1+1}(i_1) = i_1$

Look at $\{1, \dots, n\}$ and remove $\{i_1, c(i_1), \dots, c^{k_1}(i_1)\}$

c sends elements of $\{1, .., n\} \setminus \underline{\{i_1, .. \; c^{k-1}(i_1)\}}$ to elements in this set.

Repeat with the smaller set. ☒

<u>Ex</u> $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 3 & 2 & 5 & 7 & 8 & 4 & 1 & 9 \end{pmatrix} \in S_9$

$$(1\ 6\ 8)(2\ 3)(4\ 5\ 7)\cancel{(9)}$$

<u>Def</u> A cycle of length 2 is a <u>transposition</u>.

<u>Cor 37</u> Every element in S_n is a product of transpositions.

Pf: $(i_1 \ldots i_r) = (i_1\, i_r) \cdots (i_1\, i_3)(i_1\, i_2)$

$$\underbrace{\begin{matrix} i_1 \mapsto i_3 & i_1 \mapsto i_2 \\ i_3 \mapsto i_1 & i_2 \mapsto i_1 \end{matrix}}_{\begin{matrix} i_1 \mapsto i_2 \\ i_2 \mapsto i_3 \\ i_3 \mapsto i_1 \\ (i_1\ i_2\ i_3) \end{matrix}}$$