Lecture 7 - Sylow Theorems

Note Title 2/7/2008

Thm 25 redux: If X is a G-space, then

$$|X| = \sum_{\text{sum over disjoint orbits}} [G : \mathrm{Stab}_G(x)]$$

Pf $x \sim y$ if $y = g \cdot x$ some $g \in G$

$[x] = Gx$

$$X = \bigcup_{\substack{\text{disjoint}\\ \text{orbits}}} Gx \Rightarrow |X| = \sum |Gx| = \sum [G : \mathrm{Stab}_G(x)] \qquad \boxtimes$$

Cor 26: $|G| = |Z(G)| + \sum_{\substack{\text{nonconj}\\ x}} [G : C(x)]$.

$C(x)$ ← centralizer of x

Pf: G is a G-set via $c : G \to S_G$, $c(g)(h) = ghg^{-1}$

$$\mathrm{Stab}_G(h) = \{g \in G \mid ghg^{-1} = h\} = \{g \in G \mid gh = hg\}$$
$$= C(h)$$

If $h \in Z(G)$, then $ghg^{-1} = h$ $\forall g \in G \Rightarrow Gh = \{h\}$

$|Z(G)| = \sum_{h \in Z(G)} |Gh|$ and Gh is its own equivalence class.

Thm 25: $|G| = \sum_{\substack{h \text{ runs}\\ \text{through}\\ \text{disj orbits}}} [G : C(h)] = |Z(G)| + \sum_{\substack{x \text{ over}\\ \text{disjoint}\\ \text{orbits, } x \notin Z(G)}} [G : C(x)] \qquad \boxtimes$

$Gx = \{g \cdot x\}$

Thm 27: If $|G| = p \cdot k$, p a prime, then G has a subgroup of order p.

Pf: $\mathbb{Z}/p\mathbb{Z}$ has only 2 quotients: $\{e\}$ & $\mathbb{Z}/p\mathbb{Z}$

$\Rightarrow$ If X is a $\mathbb{Z}/p$-set, then $\mathbb{Z}/p \; x = \left\{ \begin{matrix} x \\ p \text{ distinct elements} \end{matrix} \right\}$

$$|\mathbb{Z}/p \; x| = \begin{cases} 1 \\ p \end{cases}$$

$$X = \{(a_1, \ldots, a_p) \mid a_1 \cdots a_p = e\} \subseteq G^p$$

g generates $\mathbb{Z}/p$, then $g \cdot (a_1, \dots, a_p) = (a_p, a_1, \dots, a_{p-1})$.

Thm 25: $|X| = \sum [\mathbb{Z}/p : \mathrm{Stab}_{\mathbb{Z}/p}(x)]$

$= a \cdot 1 + b \cdot p$

$|X| = |G|^{p-1}$, since can choose $a_1, \dots, a_{p-1}$ freely, and a_p is the inverse of their product

$\Rightarrow$ a must be divisible by p

$a \neq 0$, $(e, \dots, e) \in X$

$\rightsquigarrow$ $(x, \dots, x) \in X$ some $x \in G$, $\Rightarrow$ $x^p = e$ ▨

<u>Def</u> A group is a <u>p-group</u> if the order is a power of p.

A subgroup H of G is a <u>Sylow p-subgroup</u> if $|H| = p^n$, and $p^{n+1} \nmid |G|$. (p prime)

<u>Thm</u> <u>28</u> (Sylow Theorems)

1) p-Sylow subgroups exist & any p-subgroup of G sits inside a p-Sylow subgroup.

2) All p Sylow subgroups are conjugate.

3) # of p Sylow subgroups divides $|G|$ & is 1 mod p.

<u>Pf</u> Existence is by induction on $|G| = p^k \cdot n$, $(p, n) = 1$

if $k = 1$, done (Thm 27).

Thm 26: $|G| = |Z(G)| + \sum [G : C(g)]$

- $[G : C(g)]$ is relatively prime to p for some g: $|C(g)| = |G| / [G : C(g)]$

$\Rightarrow$ $|C(g)| = p^k \cdot n'$, $n' < n$ by induction, $C(g)$ has a p-Sylow s.g. of order p^k $\Rightarrow$ G has a p-sylow s.g of order p^k.

- $[G : C(g)]$ is divisible by p $\forall g$.

$\Rightarrow$ $|Z(G)|$ is divisible by p (reduce class eqn mod p)

Thm 27: $Z(G)$ has a s.g of order p, H & $H \trianglelefteq G$

$|G/H| = p^{k-1} \cdot n < |G|$, so by induction, G/H has a p-sylow s.g. & π_H^{-1}(of this s.g) is a p-sylow s.g of G.

$X = \{H \mid H \text{ is conjugate to } P\}$, P some fixed p-Sylow s.g.

G acts on X by conjugation. $P \in X$

$G \cdot P = \{\cancel{H} \mid H = gPg^{-1} \text{ some } g\} = X \Rightarrow$ There is only one orbit.

$|X| = [G : \mathrm{Stab}_G(P)]$

$\mathrm{Stab}_G(P) = \{g \mid gPg^{-1} = P\} \supseteq P$

$\Rightarrow [G : \mathrm{Stab}_G(P)] = |G| / |\mathrm{Stab}_G(P)| = p^k n / p^k \cdot m$ = relatively prime to p

$|X|$ divides $|G|$.

Let H be a p subgroup of G. $\Rightarrow$ H acts on X (via G's conj. action)

<u>Thm 25</u>: $|X| = \sum [H : \mathrm{Stab}_H(x)]$

H a p group $\Rightarrow [H : \mathrm{Stab}_H(x)] = \begin{cases} 1 & \mathrm{Stab}_H(x) = H \\ p^j & \mathrm{Stab}_H(x) \subsetneq H \end{cases}$.

$|H| = p^m$, $\quad |H| / |\mathrm{Stab}_H(x)| = p^m / ?$

$|X| \not\equiv 0 \mod p$, so for some $x \in X$, $[H : \mathrm{Stab}_H(x)] = 1$

$\Rightarrow x$ is gPg^{-1} some g $\Rightarrow P' = x$ is a p-Sylow s.g.

$\Rightarrow \mathrm{Stab}_H(x) = H \Rightarrow$ for all $h \in H$, $hP'h^{-1} = P'$

$\Rightarrow HP' = P'H$ as subsets of G

$\Rightarrow HP'$ is a subgroup of G.

$|HP'|$ is $p^{? \geq k}$ $\Rightarrow$ $\underset{\substack{\cup\!\shortmid \\ P'}}{|HP'|} = p^k$

$\Rightarrow HP' = P' \Rightarrow H \subseteq P'$

2) Apply this to some other P-sylow s.g.

If H is a p-group $\Rightarrow$ $H \subseteq gPg^{-1}$ some g.

Q is a p-sylow s.g $\Rightarrow$ Q is a p-group $\Rightarrow$

$Q \subseteq gPg^{-1}$ some g.

$|Q| = p^k = |gPg^{-1}| \Rightarrow Q = gPg^{-1}$. $\Rightarrow$ All p-sylow s.g are conjugate.

3) $X = \{P' \mid P' \text{ is a } p\text{-sylow s.g}\}.$

$P \leq G$ acts on X

$$|X| = \sum_i [P : \mathrm{Stab}_P(x)]$$

either 1 or a power of p.

$$|X| \equiv \sum_{\substack{x \text{ s.t.} \\ \mathrm{Stab}_P(x) = P}} [P : \mathrm{Stab}_P(x)] \mod p.$$

If $x = P'$ is a p-Sylow s.g. s.t. $\mathrm{Stab}_P(x) = P$

$$pP'p^{-1} = P' \text{ for all } p \in P$$

$\Rightarrow$ $P.P'$ is a subgroup $\Rightarrow$ $P.P' = P = P'$.

$\Rightarrow$ $|X| = 1 \mod p.$ $\square$