Lecture 6 - Permutations

Note Title 2/5/2008

Def A permutation group is any group of the form Σ_X: $\{f: X \to X \mid f \text{ is } 1\text{-}1 \text{ \& onto}\}$.
S_X

Thm 19 Any group is a subgroup of a permutation group.

Pf: $X = G$

Need an injective map $G \to S_G$

$\Rightarrow$ 1st Isom thm $\Rightarrow$ $G \cong \text{Im} \leq S_G$

$\phi: G \to S_G$

$\phi(g)(h) = gh$

$\phi(g)$ is 1-1 & onto:

$\phi(g)(h_1) = \phi(g)(h_2)$

$gh_1 = gh_2 \Rightarrow h_1 = h_2$ (left multiply by g^{-1})

Given $h \in G$, $\phi(g)(g^{-1}h) = g(g^{-1}h) = e \cdot h = h$

ϕ is injective

$\phi(g) = \text{Id}: G \to G$, then $g = e$

If $gh = h$ $\forall h \in G$, then $g = e$ True b/c e is unique.

$\Rightarrow \phi$ is injective & $G \cong \text{Im}(\phi)$. ▨

If $|G| < \infty$, then $|S_G| = |G|!$

Any homomorphism $G \to S_X$ is called a representation of G

If $G \to S_X$ is a homomorphism, X is a G-set

If $H \leq G$ is a subgroup, then G acts on G/H

$\longleftrightarrow$ have a homomorphism $G \to S_{G/H}$

$\phi_H(g)(aH) = gaH$

Prop 20 For any H, ϕ_H is a homomorphism.

<u>Pf</u>: $\phi_H(g \cdot h) = \phi_H(g) \circ \phi_H(h)$

mult in G — mult in $S_{G/H}$

def of ϕ_H as left mult.

$$\phi_H(g \cdot h)(aH) = (g \cdot h) \cdot aH = g \cdot (h \cdot aH) = \phi_H(g)(h \cdot aH)$$
$$= \phi_H(g)\big(\phi_H(h)(aH)\big) = \phi_H(g) \circ \phi_H(h)(aH) \quad \boxtimes$$

Lemma 21 The kernel of ϕ_H is the largest normal s.g of G contained in H.

<u>Pf</u>: $\ker(\phi_H)$ is always normal.

1) $\ker(\phi_H) \subseteq H$

2) If N is normal, $N \subseteq H$, then $N \subseteq \ker(\phi_H)$.

1) $g \in \ker(\phi_H) \longleftrightarrow \phi_H(g) = Id \Rightarrow \phi_H(g)(eH) = eH$

$\Rightarrow gH = H \Rightarrow g \in H$.

2) $n \in N \subseteq H$ $\qquad \phi_H(n)(gH) = ngH$

$N \trianglelefteq G \Rightarrow$ $\qquad ng = gn' \Rightarrow \phi_H(n)(gH) = gn'H$

$g^{-1}Ng = N$ $\qquad = gH \quad (N \subseteq H)$

i.e. $\phi_H(n) = Id$

$\Rightarrow n \in \ker(\phi_H)$ $\quad \boxtimes$

Assume G is finite.

Cor 22: If H is a s.g of G s.t. $|G| \nmid [G:H]!$, then G has a non-trivial normal s.g contained in H.

<u>Pf</u>: $\phi_H : G \longrightarrow S_{G/H}$; $Im(\phi_H)$ is a s.g of $S_{G/H}$

$\Rightarrow |Im(\phi_H)|$ divides $|S_{G/H}| = [G:H]!$

$$|Im(\phi_H)| = |G/\ker(\phi_H)| = |G| / |\ker(\phi_H)|$$

$$|G| / |\ker(\phi_H)| \;\Big|\; [G:H]!$$

$\Rightarrow |\ker(\phi_H)| \neq 1$ (it kills all factors of $|G|$ not in $[G:H]!$) ☒

Cor 23 If H is a s.g of G s.t. $|H|$ and $([G:H]-1)!$ are relatively prime, then H is normal.

Pf: $\mathrm{Im}(\phi_H)$ is a s.g. of $S_{G/H}$, so if N is the ker,

then $|G|/|N| = |G/N| = |\mathrm{Im}(\phi_H)| \mid [G:H]! = (|G|/|H|)!$

$$|G|/|N| = |G|/|H| \cdot |H|/|N| \;\Big|\; (|G|/|H|)!$$

$$\Rightarrow \quad |H|/|N| \text{ divides } ([G:H]-1)!$$

$(|H|, ([G:H]-1)!) = 1 \longleftrightarrow$ ie relatively prime

$\Rightarrow$ every factor of $|H|$ is a factor of $|N|$

$\Rightarrow |H| = |N| \Rightarrow H = N.$ ☒

(a,b) = greatest common divisor of a & b.

Cor 24 If p is the smallest prime dividing $|G|$, then any subgroup of index p is normal.

Pf $[G:H] = p$, and $|H|$ divides $|G|$ & is $\geq p$.

Cor 23: $|H|$ & $(p-1)!$ are relatively prime $\Rightarrow$ H is normal.

Def A *G-action* on a set X is a homomorphism

$$G \longrightarrow S_X$$

Remark: $\longleftrightarrow$ a functor $\underline{G} \longrightarrow \mathcal{S}ets$

$$\text{Obj } \{*\} \longmapsto X$$
$$\text{Mor } \{g\} \longmapsto \phi(g) \in \underset{\substack{\cup\!\!| \\ S_X}}{\mathrm{Hom}(X,X)}$$

Ex: G acts on itself:

$\phi : G \to S_G$, $\quad R : G \to S_G$

$R(g)(h) = hg^{-1}$ ← to make $R(gh) = R(g) \circ R(h)$

$c : G \longrightarrow S_G$

$c(g)(h) = ghg^{-1}$ = conjugation

$c(g)$ is a group homomorphism for all g.

$\longleftrightarrow$ an automorphism

$c(g)(ab) = g(ab)g^{-1} = g(aeb)g^{-1} = g(a(g^{-1}g)b)g^{-1}$

$= (gag^{-1})(gbg^{-1}) = (c(g)(a))(c(g)(b))$

Def The kernel of c is the center of G: $Z(G)$

$\leftrightarrow$ $\{g \in G \mid gag^{-1} = a \ \forall a\} = \{g \in G \mid ga = ag \ \forall a \in G\}$

ie $Z(G)$ is the set of elements that commute with everything.

Def If X is a G-set, $x \in X$, $(\phi : G \to S_X)$

- the orbit of x $Gx = \{\underbrace{g \cdot x}_{\phi(g)(x)} \mid g \in G\}$

- the stabilizer subgroup of x : $G(x) = \mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$

Underlying the orbits is an equivalence relation:

$x \sim y$ if $y = g \cdot x$ some g.

$[x] = Gx$

$|Gx| = [G : \mathrm{Stab}_G(x)] = |G/\mathrm{Stab}_G(x)|$

$g \cdot x \longmapsto g\,\mathrm{Stab}_G(x)$

$g \cdot x = h \cdot x \Leftrightarrow g^{-1}h \cdot x = x \Leftrightarrow g^{-1}h \in \mathrm{Stab}_G(x)$

$\Leftrightarrow [g] = [h]$ in $G/\mathrm{Stab}_G(x)$

Lemma 25: Class Equation:

$$|G| = |Z(G)| + \sum [G : \mathrm{Stab}_G(x)]$$

G acts on itself by conjugation.

Here $\mathrm{Stab}_G(x) = C(x) = \mathrm{Cent}(x) = \{g \mid gx = xg\}$.